

A realization of **LIE**'s method for solving ordinary
differential equations

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1 Basic Definition and Notations

Let $x_1, \dots, x_n$ be n independents and $u^1, \dots, u^m$ be m (analytic) functions depending on the x_i .

Let I (and J, K, L etc.) stand for multiindices $(i_1, \dots, i_n)$, $i_\nu \geq 0$.

Let f_J or $D_J f$ mean $\frac{\partial}{\partial x_1^{j_1}} \dots \frac{\partial}{\partial x_n^{j_n}} f$ with $f(x_1, \dots, x_n)$.

Let $|J| = j_1 + \dots + j_n$.

Definition 1 A termorder on the set of the u_J^i is a total ordering with $(u_K^i) < (u_L^j)$ and compatibility condition, that is if $u_K^i < u_L^j$ then $(u_K^i)_J < (u_L^j)_J$ for all J .

Example 1 $u_K^i < u_L^j$ iff

1. $\sum k_\nu < \sum l_\nu$ or
2. $\sum k_\nu = \sum l_\nu$ and $i < j$ or
3. $\sum k_\nu = \sum l_\nu$ and $i = j$ and $k_\mu < l_\mu$ and $k_\nu = l_\nu$ for $\nu > \mu$

is called a total degree order. The last half of the second condition and the third condition together define a lexicographic order.

Proposition 1 A term ordering is a well ordering.

As a short notation I write u_α for a certain u_J^i for example in the notation of a linear partial differential equation:

$$a_\alpha u_\alpha + \sum_{u_\beta < u_\alpha} a_\beta u_\beta = b$$

Definition 2 a_β (a_α) resp. u_β (u_α) resp. $a_\beta u_\beta$ ($a_\alpha u_\alpha$) is called (leading) coefficient resp. (leading) monomial resp. (leading) term of the above equation.

By abuse of notation we will speak of the position of a term in a termordering, meaning the position of its monomial in this ordering.

Definition 3 Let $e = \sum a_\alpha u_\alpha = 0$, $f = b_{\beta_0} u_{\beta_0} + \sum b_\beta u_\beta = 0$ be two lpde. Then e is reducible by f iff $u_{\alpha_0} = D_J u_{\beta_0}$ for some α_0, J (and b_{β_0} assumed unequal 0). The lpde

$$b_{\beta_0} e - a_{\alpha_0} D_J f = 0$$

is a reduction of e by f (at $a_{\alpha_0} u_{\alpha_0}$).

Definition 4 Let E, F be two sets of lpde. E is reducible by F , if there are $e \in E$, $f \in F$ with e reducible by f . Otherwise E is called reduced relative (or with respect) to F . If E is reduced relative to E it is called autoreduced. If E or F consist of a single element only, it will be allowed to use these notations with the respective elements e or f in place of E or F .

Definition 5 Let e, e' be two lpde, F a set of lpdes. e is called reduced to e' by F iff there is a sequence $e = e_0, \dots, e_r = e'$ with e_{i+1} being a reduction of e_i by a suitable $f_i \in F$. As a short notation we will write $e \xrightarrow{F} e'$.

Lemma 1 Let e be an lpde, F a set of lpde, then e can be reduced to an e' by F with e' being reduced relative to F .

Remark 1 The algorithm which effectuates this reduction shall be called $\text{reduce}(e, F)$, so that $e' = \text{reduce}(e, F)$. It is not unambiguous, one realization is:

- Algorithm 1**
1. set e_1 to e
 2. sort F into $f_1, \dots, f_r$ by order of leading term.
 3. find lowest f_i that reduces e_1 . Call it f_0
If none exists return e_1
 4. find highest term $a_\alpha u_\alpha$ in e_1 that reduces relative to f_0
 5. reduce e_1 with respect to f_0 at this term.
call result e_1
 6. goto 3.

2 Janet-Bases

2.1 Janet-Bases — Definition

Definition 6 Let $e = a_\alpha u_\alpha + r, e' = a_\beta u_\beta + r'$ with $u_\alpha = u_J^i, u_\beta = u_K^i$ be two lpde with leading monomials of the same function. Then their integrability condition is the lpde

$$a_\beta D_L e - a_\alpha D_M e'$$

in which L, M are chosen such that $D_L D_J u^i = D_M D_K u^i$ and for any L', M' that fulfill this condition too, there is a P with $D_{L'} D_J u^i = D_P D_L D_J u^i$.

We will write $\text{IGB}(e, e')$ for this integrability condition.

Definition 7 Let E be system of lpde. Then E is called passive or orthonomic, iff for all e, e' where $\text{IGB}(e, e')$ is defined,

$$\text{IGB}(e, e') \xrightarrow{E} 0$$

is fulfilled.

Definition 8 Let E be a system of lpde. The ideal of E (E) is the smallest set of lpde that:

1. contains E
2. for all $e, e' \in (E)$, $e + e' \in (E)$.
3. for all $e \in (E)$, $D_J e \in (E)$.
4. for all a from the coefficient domain, $e \in (E)$, $ae \in (E)$

Definition 9 Let E, G be two systems of lpde. Then G is called a Janet-Base of E iff $(E) = (G)$ and G is passive and autoreduced.

Theorem 1 Let e be an lpde, G be a Janet-Base. Then $e \in (G)$ iff $e \xrightarrow{G} 0$

2.2 Janet-Bases — Construction

Let E be a finite system of lpde. To compute E' with E' autoreduced and $(E) = (E')$ one can use the following algorithm:

Algorithm 2 1. set E_1 to E

2. order E_1 in increasing order of leading terms to a list $e_1, \dots, e_r$, leave out elements that are zero.

3. set i to 2.

4. if $i > r$ goto 10

5. set j to 1

6. if $j = i$ goto 9

7. try to reduce e_j by e_i . if possible do it and go to 2.

8. increase j by 1. goto 6

9. increase i by 1. goto 4

10. return E_1

Now an algorithm to compute a Janet-Base of a finite system of lpde E can be given:

Algorithm 3 1. set E_1 to E .

2. set E_1 to its autoreduction (Algorithm 2)

3. set F to the set of $\text{IGB}(e, e')$ with $e, e' \in E_1$ where $\text{IGB}(e, e')$ is defined.

4. replace every element $f \in F$ with its reduction f' by E_1 (Use Algorithm 1 to compute $f \xrightarrow{E_1} f'$). If all elements reduced to 0 then return E_1

5. set E_1 to $E_1 \cup F$. goto 2

Remark 2 JANET gave a method to do step 3 more efficiently, analogously to BUCHBERGER'S rules for unnecessary reductions but from a different point of view.

2.3 Parametric and principal derivatives

Definition 10 Let G be a Janet-base in the functions u^i , then the u_j^i obviously fall into two classes, those that are reduced by G , and the others. The former are called principal derivatives, the latter parametric derivatives. The set of parametric derivatives is also called complementary set.

Remark 3 This designation comes from the use that is made of Janet-bases in total degree order to the analysis of power series solutions of a system of lpde, and the determination of the degrees of freedom in the solution.

Proposition 2 Let G be a Janet-base, with r parametric derivatives. Then the general solution to G depends on r parameters.

Corollary 1 A diffeomorphism $y^i = \phi_i(x^1, \dots, x^n)$ doesn't change the number of parametric derivatives of a Janet-base, that is the Janet-base G' of the system of lpde E' derived from transforming a Janet-base G has the same number of parametric derivatives as has G .

3 Lie-algebras of vectorfields (lav)

3.1 Definition of an lav

In the following, vectorfields are considered on R^n resp. C^n , and in expressions like $\sum a^i \partial_i$, a^i should be considered an analytic or infinitely differentiable function of the coordinates $x^1, \dots, x^n$, with ∂_i obviously standing for $\frac{\partial}{\partial x^i}$.

Two vectorfields $X = \sum a^i \partial_i, Y = \sum b^j \partial_j$ can be composed to a third $[X, Y] = \sum c^k \partial_k$ defined by $[X, Y]f = X(Yf) - Y(Xf)$ or in coordinates:

$$c^k = \sum (a^i \partial_i b^k - b^i \partial_i a^k)$$

This is called the *bracket product* of X and Y and fulfills besides linearity in X and Y :

$$\begin{aligned} [X, Y] &= -[Y, X] && \text{Antisymmetry} \\ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] &= 0 && \text{Jacobi-Identity} \end{aligned}$$

From this it follows, that every vectorspace V of vectorfields is a lie-algebra under $[-, -]$ if it is closed under the bracket product.

An important fact is stated in the following

Proposition 3 *Let X, Y be two vectorfields defined on U , $\phi : U \rightarrow V$ be a diffeomorphism, U, V open sets in R^n (or C^n). Then $[\phi_* X, \phi_* Y] = \phi_* [X, Y]$*

which can be used to establish easily

Proposition 4 *Let X, Y be two vectorfields in U , that are tangential to a submanifold $N \subset U$. Then $[X, Y]$ is tangential to N too.*

This will be used later in analyzing symmetries of differential equations.

Another use of proposition 3 is to define the *type of an lav*:

Definition 11 *The type of an lav $\mathcal{L}$ is the set of all $\mathcal{L}'$ with $\mathcal{L}' = \phi_* \mathcal{L}$ for a diffeomorphism ϕ .*

3.2 The 27 types of lav in the plane

The list in Appendix A shows all types of lav of the plane, the form in which they are written in this list shall be called their *normal form*.

3.3 lav as solutions of systems of differential equations

In the following we will consider *finite dimensional* lav.

These are given, in the simplest case by a family of base elements, $X_1, \dots, X_r$, $X_i = \sum a_i^j \partial_j$, with explicitly known coefficients a_i^j .

This can be expressed otherwise by saying, that we have n coordinate functions $\xi_1, \dots, \xi_n$ with each $\xi_i(x^1, \dots, x^n, a_1, \dots, a_r)$ explicitly known and depending linearly on $a_1, \dots, a_r$.

It is a theorem proved by LIE, that every such system of functions can be given as the solutions of a system of lpde, that is we can find a system E of lpde whose general solution contains r free constants and coincides with the ξ_i (it is not necessary that the numbers of variables and functions are the same, as is the case here).

In the following I will call lav that are presented by lpdes *implicitly given* as opposed to those given in the way mentioned at the start of this subsection, which I call *explicitly given*.

To construct an implicit representation from an explicit one, the following algorithm can be used:

- Algorithm 4**
1. set V to the list of variables $\{x_1, \dots, x_n\}$.
 2. set E to a list of expressions $\{\Xi_1 - \xi_1, \dots, \Xi_m - \xi_m\}$ where Ξ_i are functional indeterminates depending on the variables in V .
 3. set C to the list of constants $\{a_1, \dots, a_r\}$
 4. set D to $\{\}$
 5. move all e from E that are free of C to D .
 6. if E is empty, return D .
 7. choose e from E .
 8. choose c appearing in e and C .
 9. solve e for c and write it in the form $c = e'$
 10. differentiate e by all v from V to get a list $e_1, \dots, e_n$. add this list to E .
 11. remove all appearances of c in E by substituting $c = e'$ in all elements of E .
 12. remove c from C .
 13. goto 5

3.4 Lie-algebra constructions in implicitly given lav

It is an interesting question, what information about the lie-algebra structure of an lav can be extracted from an implicit representation (finite dimension will be assumed throughout).

In the following I will show, how to

- determine the dimension of the algebra
- compute the implicit representation of the derived algebra
- compute the structure constants
- compute the implicit representation of the solvable radical

Let E be the propounded system of lpde, which can be assumed to be a Janet-base.

3.4.1 Dimension

Just count the parametric derivatives.

3.4.2 Derived algebra

For the derived algebra of an lav $\mathcal{L}$ given by a Janet-Base of lpde E , start with the formula for the bracket product:

$$\xi^{3i} = \xi^{1j} \xi_j^{2i} - \xi^{2j} \xi_j^{1i} \quad (1)$$

Let now $\alpha_1, \dots, \alpha_r$ stand for the r parametric derivatives of $\mathcal{L}$, assumed to be of dimension r (remember notational convention from the beginning of the text).

Then equation (1) allows us to express all $\xi^3_{\alpha_i}$ as function of the $\xi^1_{\alpha_i}$ and the $\xi^2_{\alpha_i}$ by performing the differentiations implicated by the α_i on (1) and doing the reductions with respect to the Janet-Base E on the derivatives of ξ^1 and ξ^2 which appear by after the differentiations.

As a result, we get expressions

$$\xi^3_{\alpha_i} = F_{\alpha_i}(\xi^1_{\alpha_j}, \xi^2_{\alpha_k}) \quad (2)$$

Now the lie algebra property (closedness under $[-, -]$) tells us, that the $\xi^3_{\alpha_i}$ have at least to fulfill all the lpde in E . If they fulfill additional relations between them, then the derived algebra $[\mathcal{L}, \mathcal{L}]$ will be smaller than $\mathcal{L}$.

So we make an ansatz for such an equation:

$$\sum c_{\alpha_i} \xi^3_{\alpha_i} = 0$$

Putting in (2) this reads as

$$\sum c_{\alpha_i} F_{\alpha_i}(\xi^1_{\alpha_j}, \xi^2_{\alpha_k}) = 0 \quad (3)$$

It must be noted that this a polynomial expression in the $\xi^1_{\alpha_j}$ and $\xi^2_{\alpha_k}$, *which can only vanish identical, if all the polynomial coefficients vanish.*

This is because $\xi^1_{\alpha_j}$ and $\xi^2_{\alpha_k}$ are *parametric* derivatives whose values can be chosen arbitrarily.

So the condition of vanishing coefficients in (3) gives us conditions *that are linear in the c_{α_i} .*

A base of the linear space of solutions provides us with the equations that we have to add to E to get a system of lpde whose solution space is the derived lav of $\mathcal{L}$

3.4.3 Structure constants

This is the most important algebraic information about an lav, and the algorithm to extract it *without solving the lpdes that define the lav* stems from REID.

It can be presented greatly simplified in notation by sticking to the finite-dimensional case, which I will do in the following.

Let the general notation be as in the last subsection, so that we take our start with E , a Janet-Base implicitly defining a lav $\mathcal{L}$, and we write down again the equation:

$$\xi^{3i} = \xi^{1j} \xi_j^{2i} - \xi^{2j} \xi_j^{1i} \quad (4)$$

and their consequence:

$$\xi^3_{\alpha_i} = F_{\alpha_i}(\xi^1_{\alpha_j}, \xi^2_{\alpha_k}) \quad (5)$$

Choose now a base $X_{\alpha_1}, \dots, X_{\alpha_r}$ of elements of $\mathcal{L}$ by defining X_{α_i} , which should stand for $X_{J_i}^{m_i}$

$$(X_{\alpha_i})_J^l = \begin{cases} 1 & \text{if } J = J_i \text{ and } l = m_i \\ 0 & \text{otherwise} \end{cases}$$

where l, J run through the possible values, that make X_J^l appear under the X_{α_i} .

That this defines a base of solutions follows from *the isomorphism between the space of parametric derivatives and the space of solutions, that is given by developing into power series with the parametric derivatives as initial data.*

It is now easy to compute $[X_{\alpha_{i_0}}, X_{\alpha_{j_0}}]$ and express the result as a linear combination of X_{α_k} .

For this, put such $\xi_{\alpha_i}^1$ into (5) that $\xi_{\alpha_{i_0}}^1$ is 1 and the others are 0 and put such $\xi_{\alpha_j}^2$ into (5) that $\xi_{\alpha_{j_0}}^2$ is 1 and the others are 0.

The result is an expression for each $\xi_{\alpha_k}^3$ as a function in the $x^1, \dots, x^n$. Call this function $c_{i_0 j_0}^k$. Then

$$[X_{\alpha_{i_0}}, X_{\alpha_{j_0}}] = \sum c_{i_0 j_0}^k X_{\alpha_k}$$

again because the values of the parametric derivatives determine the solution, and are equal on both sides of the equation.

So we have found the structure constants under the form of the system of functions c_{ij}^k which give a system of constants for each system of values $x_0^1, \dots, x_0^n$ on which they are all defined.

(It should be superfluous to remark that this construction is free of the necessity to explicitly determine the X_{α_i} defined above).

3.4.4 Solvable radical

Definition 12 A lie algebra L is called solvable, if its derived series terminates into 0, where the derived series is defined by $L^0 = L$ and $L^{i+1} = [L^i, L^i]$.

Because $I + J$ is solvable for solvable ideals I, J of L , there exists a maximal solvable ideal R of L , which is also called the *solvable radical*. and denoted $\text{Rad } L$

To compute the solvable radical the definition given above is not very useful, but another one can be given in terms of a new concept, the so called *Killing form*.

Definition 13 Let L be a lie algebra, then there is a symmetric bilinear form κ , the Killing form, on L , that is defined by

$$\kappa(X, Y) = \text{Tr}(ad X \cdot ad Y)$$

where $ad X(Y)$ is defined as $[X, Y]$.

If L is given by a base X_i and structure constants c_{ij}^k , then κ is determined by $\kappa_{ij} = \kappa(X_i, X_j)$ which can be computed to be

$$\kappa_{ij} = \sum_{p,q} c_{ip}^q c_{jq}^p$$

Now the following holds true:

Theorem 2 Let L be a finite dimensional lie algebra over a field k of characteristic 0. Then $\text{Rad } L$ is the orthogonal complement of $[L, L]$ with respect to κ , or in formula:

$$\text{Rad } L = \{X \mid \kappa(X, Y) = 0 \quad \forall Y \in [L, L]\}$$

Let $\mathcal{L}$ now be an implicitly given lav, and $\mathcal{L}^1 = [\mathcal{L}, \mathcal{L}]$ its derived algebra.

The methods of 3.4.2 bring us in the possession of a Janet-Base F for $\mathcal{L}^1$, and with the techniques from 3.4.3 we get the structure constants and therefore the Killing form κ of $\mathcal{L}$.

Let us now make an ansatz by:

$$\sum_{i,j} \kappa_{ij} \xi^1_{\alpha_i} \xi^2_{\alpha_j} = 0 \quad (6)$$

where, as practiced above, α_i, α_j should run through the parametric derivatives of $\mathcal{L}$.

In (6) $\xi^1_{\alpha_i}$ shall stand for an undetermined element of $\text{Rad } \mathcal{L}$ and $\xi^2_{\alpha_j}$ for an element from $\mathcal{L}^1$.

Now reduce the $\xi^2_{\alpha_j}$ in (6) by F , so that only parametric derivatives of $\mathcal{L}^1$ and $\mathcal{L}$ remain. Then for the expression to vanish identically it is necessary and sufficient that every coefficient of a remaining $\xi^2_{\alpha_j}$ should vanish.

But these coefficients are all lpde in $\xi^1_{\alpha_i}$ and these are the lpde that have to be added to the lpdes that determine $\mathcal{L}$ to get those that determine $\text{Rad } \mathcal{L}$.

4 Symmetry lav of differential equations

4.1 Overview

Let $x^1, \dots, x^n$ be n independents and $u^1, \dots, u^m$ be m (smooth or analytic) functions depending on the x^i .

A system of differential equations in the u^i may be given as

$$\omega_k(x^1, \dots, x^n, \dots, u^i, \dots) = 0 \quad (7)$$

where the ω_k are analytic or rational functions in their arguments.

By a field of infinitesimal displacements in the product space of x^i and u^j , each x^i is translated into $\bar{x}^i = x^i + \varepsilon \xi(x^i, u^j)$ and each u^j into $\bar{u}^j = u^j + \varepsilon \eta(x^i, u^j)$.

In this way we get new functions $\bar{u}^j(\bar{x}^1, \dots, \bar{x}^n)$.

Correspondingly the derivatives u^i_J change by an infinitesimal amount to $\bar{u}^i_J$, which we will now compute:

We start from the relation:

$$d\bar{u}^i_J = \bar{u}^i_{J_k} d\bar{x}^k \quad (8)$$

Let now the infinitesimal displacement of u^i_J to $\bar{u}^i_J$ assumed to be known and designated as $\varepsilon \eta^i_J$ then we can rewrite (8) as:

$$d(u^i_J + \varepsilon \eta^i_J) = \bar{u}^i_{J_k} d(x^k + \varepsilon \xi^k) \quad (9)$$

Now we use the relation that $df = D_k f dx^k$ for a function f depending on u^i_J 's, where D_k designates total differentiation by x^k , that is

$$D_k = \frac{\partial}{\partial x^k} + \sum_{i,J} u^i_{J_k} \frac{\partial}{\partial u^i_J}$$

Then we can rewrite (9) as:

$$u^i_{J_l} dx^l + \varepsilon D_l \eta^i_J dx^l = \bar{u}^i_{J_l} dx^l + \bar{u}^i_{J_k} \varepsilon D_l \xi^k dx^l \quad (10)$$

As this holds true for each l we can single out a coefficient of one dx^l and write (10) as:

$$\bar{u}_{J_l}^i - u_{J_l}^i = \varepsilon (D_l \eta_J^i - \bar{u}_{J_k}^i D_l \xi^k) \quad (11)$$

As $\bar{u}_{J_k}^i$ differs from $u_{J_k}^i$ only by an infinitesimal amount, we can replace the former by the latter on the right side of (11):

$$\bar{u}_{J_l}^i - u_{J_l}^i = \varepsilon (D_l \eta_J^i - u_{J_k}^i D_l \xi^k) \quad (12)$$

And now we arrived at our conclusion: The infinitesimal amount, by which $\bar{u}_{J_l}^i$ differs from $u_{J_l}^i$ is

$$\varepsilon (D_l \eta_J^i - u_{J_k}^i D_l \xi^k)$$

in other words

Proposition 5 *The infinitesimal displacements of u_J^i resulting from an infinitesimal point-transformation are given by the following recursion relation:*

$$\eta_{J_l}^i = D_l \eta_J^i - u_{J_k}^i D_l \xi^k \quad (13)$$

Let us now use the following

Definition 14 *Be x^i, u^j be as above. The space $\mathcal{J}^k$ with coordinates x^i, u^j, u_J^i , where $|J| \leq k$ is called the k -th order jet space associated to the space of x^i, u^j .*

Then the extension of a field of displacements x^i, η_J^j in $\mathcal{J}^k$ to a field of displacements in $\mathcal{J}^l$ (with $l > k$) using the formula from Proposition 5 is called *prolongation*.

If $X^{(k)}$ designates a vectorfield in $\mathcal{J}^k$ space, then we will designate its l -th prolongation by $X^{(k)(l)}$. It is a vectorfield in $\mathcal{J}^{k+1}$.

It is interesting to note, that $X^{(k)(1)} = X^{(k+1)}$ in the sense, that considering all the u_J^i of $\mathcal{J}^k$ just as ordinary variables u^{iJ} and doing a first prolongation is the same as applying the formula of proposition 5 and interpreting u_k^{iJ} as $u_{J_k}^i$.

For the theory of lav it is important that the following proposition holds:

Proposition 6 *Let $X^{(k)}, Y^{(k)}$ be vectorfields in $\mathcal{J}^k$. Then*

$$\left[X^{(k)}, Y^{(k)} \right]^{(l)} = \left[X^{(k+l)}, Y^{(k+l)} \right]$$

From this it follows immediately that *an lav transforms into an lav under prolongation*.

With this machinery in hands, we can define the notion of a system of differential equations invariant under a point transformation.

Let us assume that the system of deq

$$\omega_k(x^1, \dots, x^n, \dots, u_J^i, \dots) = 0$$

cuts out a submanifold N in $\mathcal{J}^l$. Then we will call the deq's ω^k *invariant under the vector field X of point-transformations* if $(X^{(k)} \omega^p)(P) = 0$ for all P on N , or geometrically if $X^{(k)}$ is tangential to N .

From proposition 4 we see that the set of vectorfields X which leave a given set of deq invariant *must be a lav* (which can be $\mathbf{0}$ of course).

After all this, the question, how the lav of symmetries of a set of deq can be computed can be answered rapidly.

- start with functional indeterminates $\xi^p(x^j, u^k), \eta^q(x^j, u^k)$ and prolong the vectorfield $X = \xi^p(x^j, u^k) \partial_{x^p} + \eta^q(x^j, u^k) \partial_{u^q}$ to $X^{(k)} = \xi^p \partial_{x^p} + \eta^q \partial_{u^q} + \eta_J^i \partial_{u_J^i}$.

- form the equations

$$e^p = X^{(k)}\omega^p(x^1, \dots, x^n, \dots, u_J^i, \dots) = 0$$

- reduce the equations e^p by the system of ω^p . This can be done for example by solving each ω^p for some $u_{J_p}^{i_p}$, putting it into the form of $u_{J_p}^{i_p} = \Omega^p(x^1, \dots, x^n, \dots, u_J^i, \dots)$ and substituting $\Omega^p(x^1, \dots, x^n, \dots, u_J^i, \dots)$ in the e^p for $u_{J_p}^{i_p}$.
- the result is a set of equations

$$F^p(x^1, \dots, x^n, \dots, u_J^i, \dots, \xi_J^j, \dots, \eta_K^i, \dots) = 0$$

which are *polynomial in the u_J^i and lpde's in the ξ_J^j, η_K^i .*

- take now the coefficients of the F^p with respect to the u_J^i ($|J| > 0$). They are lpde's in the ξ_J^j, η_K^i and *these are the lpde's that describe the lav of symmetries of the ω^p .*

4.2 Determination of ODEs with given symmetry lav

For the methods in the next section to be applicable it will be necessary, to know beforehand, what lav can possibly occur as lav of odes of a given maximal order.

An important partial answer is given by the following theorem

Theorem 3 *An ode of order $n \geq 3$ has a symmetry lav of dimension at most $n+4$, an ode of order 2 has an lav of dimension at most 8.*

With the help of this theorem it is possible to select from the list of lav of the plane a finite number of lav which have a dimension that is allowed by the theorem above.

But in general some of these lav will occur only as symmetry lav of odes of larger orders than initially presumed, and therefore must be found and excluded.

This is done by the method of differential invariants.

Definition 15 *Let $\mathcal{L}$ be an lav of x^i, u^j space and $\mathcal{L}^{(k)}$ be its prolongation as an lav of the jet space. Then a function $G(x^i, u^j, \dots, u_J^k, \dots)$ on the jet-space is called a differential invariant of $\mathcal{L}$ if*

$$X^{(k)}G = 0$$

for all $X^{(k)}$ in $\mathcal{L}^{(k)}$.

By this definition, a differential equation, having a certain lav $\mathcal{L}$ as symmetry lav is nothing else but a differential invariant of this lav.

It is obvious from the definition, that for an explicitly given lav $\mathcal{L}$ and a given order k, the explicit determination of the differential invariants of order k is identical to the solution of a system of linear partial differential equations for one function ($G(x^i, u^j, \dots, u_J^k, \dots)$). These lpde can be chosen to be those that arise from prolonging the elements of a base of $\mathcal{L}$.

The following proposition is just a way to say that every solution of a system of lpde is function of a base of solutions:

Proposition 7 *Let $\mathcal{L}$ be an lav and $\psi_1, \dots, \psi_s$ be a base of invariants of $\mathcal{L}$ of order k. Then for every differential equation ω of order k there is a function F with*

$$\omega = F(\psi_1, \dots, \psi_s)$$

In the special case of an ode we know, that

- x, u space has dimension 2
- therefore k jet-space has dimension $k + 2$.

Now an lav of dimension r will give rise to r lpde which G has to fulfill, and therefore, in general, the first jet-space in which an invariant will exist will be $r - 1$ jet-space. We will call this invariant ϕ_1 . In r jet-space we will then have 2 differential invariants, one of which will be ϕ_1 , the other we will call ϕ_2 . Because every function of invariants is an invariant, ϕ_1, ϕ_2 are not uniquely determined, they are understood as a base of functional independent solution to the system of lpde.

Additionally we have the following theorem, which greatly simplifies the task of finding invariants:

Theorem 4 *Let $\mathcal{L}$ be an lav of order r and ϕ_1, ϕ_2 its two basic differential invariants of order $r - 1$ and r respectively. Then a base of differential invariants of order $\leq r + k$ is given by*

$$\phi_1, \quad \phi_2, \quad \frac{d\phi_2}{d\phi_1}, \quad \dots \quad \frac{d^k \phi_2}{d\phi_1^k}$$

5 Determination of the symmetry lav type of an ODE

5.1 The problem exemplified

Let a differential equation

$$y'' y - \frac{1}{2} y^2 - 2 = 0 \tag{14}$$

be given.

We compute the lpdes determining their symmetry lav as:

$$\begin{aligned} & \eta_{xx} \\ & \eta_y - \frac{\eta}{y} \\ & \xi_y \\ & \xi_x - \frac{\eta}{y} \end{aligned}$$

Now we do a coordinate transformation $x = x/y, y = x y$ which gives for the differential equation

$$\begin{aligned} & 2 y^8 x^4 \left(\frac{\partial^2}{\partial x^2} y \right) + (-2 y^6 x^5 + 2 x^3) \left(\frac{\partial}{\partial x} y \right)^3 \\ & + (-2 y^7 x^4 - 6 y x^2) \left(\frac{\partial}{\partial x} y \right)^2 + (4 y^8 x^3 + 6 y^2 x) \left(\frac{\partial}{\partial x} y \right) \\ & - 2 y^3 \end{aligned}$$

and for the symmetry lav

$$\begin{aligned} \xi_{xx} - 4 \frac{\xi_x}{x} + 6 \frac{\eta}{x y} + 6 \frac{\xi}{x^2} \\ \eta_y + \xi_x - 4 \frac{\eta}{y} - 2 \frac{\xi}{x} \\ \eta_x - \frac{y \xi_x}{x} + 3 \frac{\eta}{x} + 2 \frac{y \xi}{x^2} \\ \xi_y + \frac{x \xi_x}{y} - \frac{\xi}{y} \end{aligned}$$

If we want to recognize, that the *types* of the symmetry lav of both differential equations are equal, how can we proceed ?

5.2 The method of indeterminate transformations

A method to get a complete overview over the possibilities for Janet-bases for a certain set of lav is:

Do the following for each lav $\mathcal{L}$ in the list:

- compute the Janet-base E of the *normal form* of $\mathcal{L}$.
- apply a coordinate transformation

$$\begin{aligned} x &= \sigma(x, y) \\ y &= \rho(x, y) \end{aligned}$$

with indeterminate $\sigma(x, y)$, $\rho(x, y)$ to E and get a system of lpde E' .

- compute a Janet-base E_1 of E' .
- the coefficients in the lpde of E_1 are rational differential polynomials in σ , ρ , σ_x , ρ_x , σ_y , ρ_y , σ_{xx} , $\dots$

If these coefficients of E_1 contain denominators $d_1, \dots, d_r$, that could possibly vanish, recalculate for each d_i a new Janet-Base E_{d_i} from E' by assuming $d_i = 0$ throughout the calculation.

Remark: In most cases these d_i are just one of the four σ_x , σ_y , ρ_x , ρ_y .

If coefficients in the E_{d_i} contain again denominators $d_{i1}, \dots, d_{ir_i}$ recalculate again, with $d_i = 0, d_{iji} = 0$ as assumptions.

Repeat this process until all denominators in all cases considered are provably not zero.

In most cases it will be sufficient to consider the first level (denominators in E_1) only.

Remark 4 *The number of independent variables need of course not be 2 as above described. It could be any number $x^1, \dots, x^n$ and the number 2 was only chosen for ease of notation (and because all the lav appearing for ode are lav in the plane).*

Definition 16 *The list of the $E_1, E_{d_i}, \dots$ flowing from the above calculation will be called the Janet-base pattern list belonging to the original list of lav in normal form.*

Such a pattern list will give a first mean to distinguish different types of lavs by their Janet-base:

Definition 17 The type of a Janet–base E consisting of lpdes $e_1, \dots, e_r$ is the set of leading monomials $\xi_{\alpha_1}, \dots, \xi_{\alpha_r}$ of $e_1, \dots, e_r$.

If we now have as given a Janet–base of an unknown lav from our list, then we are able to conclude, that it can only belong to those lav(s), which give rise to a Janet–base of the same type under the above method.

This will in general be more than one, so that we have to employ further techniques to finally decide from which normal form lav our lav came about. But if we postpone this problem a moment and just consider it as possible, we see at once from the above method of calculation of the pattern list, that we will get the additional information whether some expression d_i in the $\sigma, \rho, \sigma_x, \rho_x, \sigma_y, \rho_y, \sigma_{xx}, \dots$ vanishes in our given case.

So we would have determined not only the type of our lav, but additionally a property of the transformation $x = \sigma(x, y), y = \rho(x, y)$ from the normal form to our given form the lav.

The next page illustrates this for the list of lavs that occur as 3 dimensional symmetry lavs of third order ordinary differential equations.

The left side of the table contains types of Janet–bases, the right side the corresponding lavs (the additional letters ffff, tfff, etc. code for information about $\sigma_x, \sigma_y, \rho_x, \rho_y$ in this order. A letter t means that the corresponding derivative is zero).

$\eta_{xx}, \eta_y, \xi_y, \xi_x$	$g_{3,17,3,ftff}, g_{3,17,4,ftff}$ $g_{3,17,1,ftff}, g_{3,5,ftff}$ $g_{3,13,ftff}, g_{3,17,2,ftff}$
$\eta_{yy}, \eta_x, \xi_y, \xi_x$	$g_{3,5,fttf}$
$\xi_{xx}, \eta_y, \eta_x, \xi_y$	$g_{3,13,ffff}, g_{3,17,3,ffff}$ $g_{3,17,4,ffff}, g_{3,17,1,ffff}$ $g_{3,5,ffff}, g_{3,10,ffff}$ $g_{3,20,ffff}, g_{3,24,ffff}$ $g_{3,7,ffff}, g_{3,17,2,ffff}$
η_{yyy}, η_x, ξ	$g_{3,8,fttf}$
ξ_{xxx}, ξ_y, η	$g_{3,15,ffff}, g_{3,8,ffff}$
η_{xxx}, η_y, ξ	$g_{3,15,ftff}, g_{3,8,ftff}$
$\xi_{yy}, \eta_y, \eta_x, \xi_x$	$g_{3,17,3,tfff}, g_{3,17,4,tfff}$ $g_{3,17,1,tfff}, g_{3,5,fftf}$ $g_{3,10,ffff}, 1, g_{3,20,fftf}, 1$ $g_{3,20,ffft}, 1, g_{3,20,ffff}, 1$ $g_{3,7,ffff}, 1, g_{3,13,tfff}$ $g_{3,17,2,tfff}$
$\eta_{yy}, \eta_{xy}, \eta_{xx}, \xi$	$g_{3,16,ftff}$
ξ_{yyy}, ξ_x, η	$g_{3,15,tfff}, g_{3,8,fftf}$
$\xi_{yy}, \xi_{xy}, \xi_{xx}, \eta$	$g_{3,16,ffff}$

5.3 Lie's invariants λ and ν

Let us assume in the following, that the lav is presented implicitly as a set of lpde in $\xi(x, y), \eta(x, y)$, with $\xi \partial_x + \eta \partial_y$ as vectorfields.

LIE noticed, that by a transformation of coordinates $x = \sigma(x, y), y = \rho(x, y)$ the following two expressions change as follows:

$$\xi_x + \eta_y + \Omega(0) \quad (15)$$

reproduces identically and

$$\mu^2 \xi_y - \mu \nu (\xi_x - \eta_y) - \nu^2 \eta_x + \Omega(0) \quad (16)$$

becomes

$$\mu_1^2 \xi_y - \mu_1 \nu_1 (\xi_x - \eta_y) - \nu_1^2 \eta_x + \Omega(0) \quad (17)$$

with

$$\begin{aligned} \mu_1 &= \rho_x \mu + \sigma_x \nu \\ \nu_1 &= \sigma_y \mu + \rho_y \nu \end{aligned} \quad (18)$$

where $\Omega(0)$ signifies an otherwise undetermined expression in x, y, ξ, η , linear in ξ, η .

Lemma 2 *Let $\mathcal{L}$ be an lav, implicitly given by a Janet-base E consisting of lpde $e_1, \dots, e_r$. If (15) reduces to $\Omega(0)$ by E , then the same will be true for all lav $\mathcal{L}'$ of the same type as $\mathcal{L}$.*

Proof: Let

$$\xi_x + \eta_y = 0 \mod (E, \Omega(0))$$

that is

$$\xi_x + \eta_y = \sum r_j D_{J_j} e_{n_j} \mod \Omega(0)$$

After a coordinate transformation it follows that

$$\xi_x + \eta_y = \sum r'_k D_{J_k} \phi^* e_{n_k} \mod \Omega(0)$$

where the transformed lpde $\phi^* e_{n_k}$ are reduced to 0 modulo the Janet-Base E' by which $\mathcal{L}'$ is implicitly given. Therefore

$$\xi_x + \eta_y = 0 \mod (E', \Omega(0))$$

So the vanishing of $\xi_x + \eta_y$ is an lav type invariant, which LIE called λ and assigned it 1 if vanishing occurs and 0 otherwise.

The invariant, which can be extracted from the invariance of (16) under coordinate transformation is the number of pairs $(\mu(x, y), \nu(x, y))$ different from $(0, 0)$ and linear independent (over C) from each other, that, when entered into (16) make (16) reducible to $\Omega(0)$ by E (where E denotes again the Janet-base belonging to $\mathcal{L}$)

Lemma 3 *Let $\mathcal{L}$ be an lav and E the Janet-base for its implicit representation. Then the number of linear independent $(\mu(x, y), \nu(x, y))$ different from $(0, 0)$ that make (16) reducible to $\Omega(0)$ by E can be 0, 1, 2 or infinite, which means that (16) is reducible to $\Omega(0)$ by E for any choice of $\mu(x, y), \nu(x, y)$.*

Following LIE, this invariant number is called ν .

Furthermore this number is invariant under diffeomorphic transformation of $\mathcal{L}$ and therefore an invariant of the type of an lav.

Proof: The first statement is a statement about projective solutions of a system of quadratic forms, the second follows from the fact that the transformation matrix on the right side of (18) is nonsingular.

Actually it would be enlightening to prove, that every independent solution $(\mu(x, y), \nu(x, y))$ occuring in Lemma 3 gives rise to a *system of imprimitivity* of $\mathcal{L}$ in the form of a vectorfield $\nu(x, y) \partial_x + \mu(x, y) \partial_y$, but I will refer for this to [1].

Example 2 *Let us compute λ and ν for our Janet-bases from the previous subsection.*

The expression $\xi_x + \eta_y$ reduces

- modulo the first Janet–base to

$$\frac{2}{y}\eta$$

- modulo the second Janet–base to

$$\frac{4}{y}\eta + \frac{2}{x}\xi$$

We conclude λ equals 1 (in both cases as expected).
The expression $\mu^2 \xi_y - \mu\nu(\xi_x - \eta_y) - \nu^2 \eta_x$ reduces

- modulo the first Janet–base to

$$-\frac{\mu\nu}{y}\eta - \nu^2 \eta_x$$

- modulo the second Janet–base to

$$\begin{aligned} & \frac{(\mu^2 x^2 + 2\mu\nu yx + 2\nu^2 y^2)\xi}{y x^2} + \frac{\nu(3\nu y + 4\mu x)\eta}{x y} \\ & - \frac{(\mu x + \nu y)^2 \xi_x}{x y} \end{aligned}$$

Obviously in both cases ν equals 1.

The application of these two invariants in the classification problem is to derive conditions on the coefficients of a Janet–base in a pattern–list of Janet–bases (see previous subsection), where one Janet–base pattern belongs to several possible lavs, which have different values for ν and λ .

These conditions serve as readily to compute expressions in the coefficients of the propounded Janet–base whose being zero or nonzero decides on the type of the lav represented by the propounded Janet–base.

Example 3 The Janet–base pattern to which our second Janet–base of the previous subsection belongs, is

$$\begin{aligned} & \xi_{xx} + a_1 \xi_x + a_2 \eta + a_3 \xi \\ & \eta_y + b_1 \xi_x + b_2 \eta + b_3 \xi \\ & \eta_x + c_1 \xi_x + c_2 \eta + c_3 \xi \\ & \xi_y + d_1 \xi_x + d_2 \eta + d_3 \xi \end{aligned}$$

From the groups that come into question for a second order ode, it is the pattern for $g_{3,7,\text{fff}}$, $g_{3,10,\text{fff}}$, $g_{3,13,\text{fff}}$, $g_{3,20,\text{fff}}$

They have

lav	λ	ν
$g_{3,7,\text{fff}}$	0	2
$g_{3,10,\text{fff}}$	1	2
$g_{3,13,\text{fff}}$	1	1
$g_{3,20,\text{fff}}$	0	1

We note at this point, that our Janet–bases from above are belonging to $g_{3,13,\text{fff}}$, which would of course be trivial for the first one because it was in normal form coordinates, but definitely not for the second.

This said, let us proceed with deriving explicit conditions in the coefficients of the Janet-base pattern:

$\xi_x + \eta_y$ reduces to

$$-b_3 \xi - b_2 \eta + (1 - b_1) \xi_x$$

Therefore $b_1 = 1$ is the condition for λ equal 1.

$\mu^2 \xi_y - \mu \nu (\xi_x - \eta_y) - \nu^2 \eta_x$ reduces to

$$\begin{aligned} &(\nu^2 c_3 - \mu^2 d_3 - \mu \nu b_3) \xi + (\nu^2 c_2 - \mu^2 d_2 - \mu \nu b_2) \eta \\ &+ (-\mu^2 d_1 + (-1 - b_1) \nu \mu + \nu^2 c_1) \left(\frac{\partial}{\partial x} \xi \right) \end{aligned}$$

Because only one quadratic form (coefficient of ξ_x) arose as condition, only the cases ν equals 1, 2 or infinite could occur. ν being infinite would necessitate $d_1 = 0$, $c_1 = 0$, $b_1 = -1$. ν being 1 necessitates vanishing of the discriminant of the quadratic form which means $4d_1 c_1 + (-1 - b_1)^2 = 0$.

So we have derived explicit conditions in the coefficients of the Janet-base.

5.4 The characteristic polynomial as a mean to distinguish lav types

For some lav the methods considered above for distinguishing the type do not suffice, because they share the same type of Janet-base and the same invariants λ and ν .

For these lav a new type of invariant can be utilized, that stems from the theory of lie algebras, namely the *characteristic polynomial of a lie algebra*.

Definition 18 Let L be a lie algebra with structure constants c_{ij}^k . Then the characteristic polynomial χ_L is defined as the characteristic polynomial of the matrix

$$a_j^k = \sum_i t_i c_{ij}^k$$

where the t_i are a set of auxiliary indeterminates.

Remark 5 Another way would have been to choose a base $X_1, \dots, X_n$ of L and introduce χ_L as

$$\det \left(\lambda \mathbf{E} - \mathbf{ad} \left(\sum_i t_i X_i \right) \right)$$

As the structure constants transform like a tensor with respect to the position of their indices, any change of base just results in a linear transformation of the t_i .

Therefore invariants of a lie algebra can be derived from its characteristic polynomial, for example the number of different factors and the degree of the factors are invariants.

Example 4 Consider the following table of characteristic polynomials which are computed from the lie algebras belonging to the lav of the same index (and are therefore indexed with the letter "l").

$l_{3,17,1}$	$-(\lambda - t_3)(-\lambda + t_3 a) \lambda$
$l_{3,17,2}$	$(\lambda - t_3)^2 \lambda$
$l_{3,17,3}$	$\lambda^2 (\lambda - t_3)$
$l_{3,17,4}$	λ^3

Obviously a linear change of t_1, t_2, t_3 doesn't change the way, these polynomials factor, and so the factorization can be used to discriminate the lie algebras.

Example 5 *If we compute the characteristic polynomial of the lav considered above given in implicit presentation, making use of the algorithm to determine the structure constants, we get the following result:*

$g_{3,17,1,ffff}$	$(\sigma_y t_3 - \lambda + \sigma_x t_2)(a \sigma_x t_2 + \sigma_y t_3 a - \lambda) \lambda$
$g_{3,17,2,ffff}$	$(\sigma_y t_3 - \lambda + \sigma_x t_2)^2 \lambda$
$g_{3,17,3,ffff}$	$-(\sigma_y t_3 - \lambda + \sigma_x t_2) \lambda^2$
$g_{3,17,4,ffff}$	λ^3

Considering that we can assume σ_x and σ_y to be not zero (and $a \neq 1$) this separates the four lav.

It would also, as in the last section, be possible to start the computation with a Janet-base pattern so that the coefficients of the characteristic polynomial become expressions in the coefficients of the pattern Janet-base (and their derivatives), but in the cases of lav of larger dimension than 4 this seems to be disadvantageous if weighed against the direct computation with the actual given Janet-base.

6 The point-transformation of an lav to normal form

6.1 An example problem

Let us remember the ode

$$\left(\frac{\partial^2}{\partial x^2} y\right) - \frac{2}{y^3} \quad (19)$$

that was transformed by $x = x/y$, $y = 1/\sqrt{2} x^2 y^2$ into

$$\begin{aligned} & 2 y^8 x^4 \left(\frac{\partial^2}{\partial x^2} y\right) + (-2 y^6 x^5 + 2 x^3) \left(\frac{\partial}{\partial x} y\right)^3 \\ & + (-2 y^7 x^4 - 6 y x^2) \left(\frac{\partial}{\partial x} y\right)^2 + (4 y^8 x^3 + 6 y^2 x) \left(\frac{\partial}{\partial x} y\right) \\ & - 2 y^3 \end{aligned}$$

Now the symmetry algebra of this new ode has the form

$$\begin{aligned} & \xi_{xx} - 4 \frac{\xi_x}{x} + 6 \frac{\eta}{x y} + 6 \frac{\xi}{x^2} \\ & \eta_y + \xi_x - 4 \frac{\eta}{y} - 2 \frac{\xi}{x} \\ & \eta_x - \frac{y \xi_x}{x} + 3 \frac{\eta}{x} + 2 \frac{y \xi}{x^2} \\ & \xi_y + \frac{x \xi_x}{y} - \frac{\xi}{y} \end{aligned} \quad (20)$$

The solution to the problem of integration would be greatly furthered if one could determine from (20) the transformation $x = x/y$, $y = 1/\sqrt{2} x^2 y^2$ without integrating this set of lpde first.

6.2 The basic system of nonlinear PDEs

Let E be any propounded Janet-base describing any lav $\mathcal{L}$, not necessary coming from a symmetry problem.

Let it be known that this lav is one from a finite set of lav that we know beforehand.

Then with the methods of the section 5 we can determine that lav from this finite set to which $\mathcal{L}$ is equivalent and from the list of Janet-base patterns computed by the method of 5.2 we can select the single one fitting our propounded Janet-base E .

So we have two Janet-bases of identical composition:

- E with equations

$$\xi_{\alpha_i} + \sum a_{\alpha_{ij}} \xi_{\alpha_{ij}} \quad (21)$$

with $a_{\alpha_{ij}}$ depending on the independent variables x, y .

- the pattern Janet-base with equations

$$\xi_{\alpha_i} + \sum a'_{\alpha_{ij}} \xi_{\alpha_{ij}} \quad (22)$$

with $a'_{\alpha_{ij}}$ depending on the indeterminate functions σ, ρ , and their derivatives $\sigma_x, \rho_x, \sigma_y, \rho_y, \sigma_{xx}, \dots$ (and maybe some additional constants from the lav).

Remark 6 *As remarked earlier the number of independent variables need not equal 2*

Now the decisive step is to compare coefficients in (21) and (22) to get a list of partial differential equations for $\sigma(x, y), \rho(x, y)$ that are nonlinear in general.

$$a_{\alpha_{ij}}(x, y) = a'_{\alpha_{ij}}(\sigma, \rho, \sigma_x, \dots) \quad (23)$$

The solutions to these pde give the transformation to normal coordinates.

This is of course no surprise, and the real problem will be to solve these nonlinear pde. For this purpose LIE invented an ingenious method, that I will present in the next subsection.

6.3 Lie's method of invariants

The idea of LIE was to prove, that certain expressions in σ, ρ and their derivatives are equal to functions of x, y alone, if one takes the equations of (23) into account.

Typically these equalities more or less immediately lead to the integration of (23), by reducing the integration work to quadratures or the integration of Riccati-equations.

For example, if one is able to find explicitly functions $f(x, y), g(x, y)$ with

$$\begin{aligned} \frac{\sigma_x}{\sigma} &= f(x, y) \\ \frac{\sigma_y}{\sigma} &= g(x, y) \end{aligned} \quad (24)$$

the function $\log \sigma$ and therefore σ can be computed by quadratures alone and indeed, for quite a number of lav the expressions on the left side of (24) allow to be written as functions of x and y alone.

How to exhibit practically those functions $f(x, y), g(x, y)$ if the equations (24) don't occur more or less explicitly under the equations of (23) will be described in the next subsection. First I will describe how to exhibit those useful expressions in σ and ρ

6.3.1 Finding LIE's invariants

Definition 19 Let $\mathcal{L}$ be a finite lav with a base $X_1, \dots, X_n$ of vectorfields. Then the lav $\mathcal{L}'$ consisting of all Y with

$$[X_i, Y] = c_{i1} X_1 + \dots + c_{in} X_n$$

with $c_{ij} \in R$ is called the inverse lav of $\mathcal{L}$.

Remark 7 To my knowledge LIE didn't give a name to this lav, so the name is by me. In modern terminology, it would be the normalizer of $\mathcal{L}$ in the (infinite dimensional) lav of all vectorfields of the plane. That it is closed under $[\cdot, \cdot]$ follows from Jacobi's identity.

Let us for the moment study the following construction: An lav $\mathcal{L}$ in the plane with coordinates x, y can be extended to an lav in any number of variables, say u, v, x, y by the obvious embedding. Let such an extension be called $\mathcal{L}^{\text{ext}}$.

It is furthermore conceivable to think of x, y as depending on u, v , which are thought as independents.

Then by the usual formalism, one can prolong the lav $\mathcal{L}^{\text{ext}}$ into the jet-space with coordinates $u, v, x, y, x_u, x_v, \dots$ to obtain a prolonged lav $\mathcal{L}^{\text{ext}(k)}$ acting on this jet-space and having elements that are generically written as

$$\xi \partial_x + \eta \partial_y + \eta_{x_u} \partial_{x_u} + \dots$$

It is then clear, what invariants under $\mathcal{L}^{\text{ext}}$ are:

Definition 20 A function $G(x, y, x_u, x_v, y_u, y_v, x_{uu}, \dots)$ is an invariant under $\mathcal{L}^{\text{ext}(k)}$, when for all $X \in \mathcal{L}$

$$X^{\text{ext}(k)} G = 0$$

Now LIE (more or less) proved the following remarkable theorem:

Theorem 5 Let $\mathcal{L}$ be an lav in normal form, $\mathcal{L}_1$ be its inverse lav. Let $G(x, y, x_u, x_v, y_u, y_v, x_{uu}, \dots)$ be an invariant of $\mathcal{L}_1^{\text{ext}(k)}$ in the previously explained sense.

Then the expression in $\sigma, \rho, \sigma_x, \sigma_y, \rho_x, \rho_y, \sigma_{xx}, \dots$ $G(\sigma, \rho, \sigma_x, \sigma_y, \rho_x, \rho_y, \sigma_{xx}, \dots)$ (given by substituting σ for x , ρ for y , x for u , y for v) can be expressed as a function of x, y in the way discussed at the entrance of this section and all expressions with this property arise as invariants in this way.

Remark 8 Stated in this form, the system of lpde (23) that implicitly enters into the theorem is the one that results from starting with $x = \sigma(x, y)$, $y = \rho(x, y)$ as the indeterminate transformation, or in another formulation, one makes the assumptions that $\sigma_x, \sigma_y, \rho_x, \rho_y$ are all unequal zero. If this can't be assumed, the coordinates of the jet-space, that would transform to zero, when the substitution of the theorem was made, have to be left out from the arguments of G .

6.3.2 An example application of LIE's invariants

We consider as an example the lav $g_{3,13}$ with $\sigma_x, \sigma_y, \rho_x, \rho_y$ assumed unequal zero.

A base of $g_{3,13}$ is

$$p, \quad xp + yq, \quad x^2p + 2xyq$$

a base of its inverse lav, that we will call $\mathcal{L}_1$, is

$$p, \quad xp, \quad yq, \quad x^2p + 2xyq$$

If we compute from these the extension and prolongation as described in the last subsection, we get a new lav $\mathcal{L}_1^{\text{ext}(k)}$ with base

$$\begin{aligned}
X_1^{\text{ext}} &= \partial_x \\
X_2^{\text{ext}} &= x \partial_x + x_u \partial_{x_u} + x_v \partial_{x_v} + x_{uu} \partial_{x_{uu}} + \cdots \\
X_3^{\text{ext}} &= y \partial_y + y_u \partial_{y_u} + y_v \partial_{y_v} + y_{uu} \partial_{y_{uu}} + \cdots \\
X_4^{\text{ext}} &= x^2 \partial_x + 2xy \partial_y + 2x_u x \partial_{x_u} + 2x_v x \partial_{x_v} + \\
&\quad (2x_u y + 2y_u x) \partial_{y_u} + (2x_v y + 2y_v x) \partial_{y_v} + \cdots
\end{aligned} \tag{25}$$

Therefore invariants of $\mathcal{L}_1^{\text{ext}(k)}$ are

$$\begin{aligned}
k &= \frac{\sigma_x}{\sigma} \\
j &= \frac{2\sigma_x \sigma_{xxy} - 4\sigma_x \sigma_{xx} \sigma_{xy} + \sigma_y \sigma_{xx}^2}{\sigma_x^3} \\
\Omega &= \frac{\sigma_{xxx}}{\sigma_x} - \frac{3\sigma_{xx}^2}{2\sigma_x^2} \\
h_1 &= \frac{\rho_x}{\rho} - \frac{\sigma_{xx}}{\sigma_x} \\
h_2 &= \frac{\rho_y}{\rho} - \frac{\sigma_{yy}}{\sigma_y}
\end{aligned} \tag{26}$$

It is obvious that ρ can be computed from h_1 and h_2 by quadratures alone, provided that σ is known.

To compute σ we use the auxiliary expression $U = \sigma_{xx}/\sigma_x$, which fulfills the two equations

$$\begin{aligned}
U_x &= \frac{1}{2}U^2 + \Omega \\
U_y &= \frac{1}{2}kU^2 + k_x U + \frac{1}{2}j
\end{aligned} \tag{27}$$

By the substitution $x = x_0 + x_1$, $y = y_0 + x_1 y_1$ (where x_0, y_0 are constants so that the right sides of the above two Riccati-equations are welldefined for $x = x_0, y = y_0$) these can be reduced to a single Riccati-equation in x_1 with y_1 as a parameter, so that the unavoidable integration work boils down to one Riccati-equation in one variable.

To determine σ from U , no integration is necessary, it suffices to differentiate and eliminate. To follow LIES argument, let

$$\Phi(U, x, y) = \text{const.}$$

be an implicit form for U which resulted from integration of the Riccati-equation above. Differentiating the left side with X_4^{ext} gives

$$X_4^{\text{ext}} \Phi(U, u, v) = \Phi_U(U, u, v) (X_4^{\text{ext}} U)$$

and with $X_4^{\text{ext}} U = 2x_u$ (remember $U = x_{uu}/x_u$)

$$X_4^{\text{ext}} \Phi(U, u, v) = 2\Phi_U(U, u, v) x_u$$

another differentiation with X_4^{ext} gives

$$X_4^{\text{ext}} X_4^{\text{ext}} \Phi(U, u, v) = 4\Phi_{UU} x_u^2 + 4\Phi_U x_u x$$

because $X_4^{\text{ext}} x_u = 2x_u x$.

Now a theorem by LIE tells us, that to determine x (that is σ) it is enough to set those three expressions Φ , $X_4^{\text{ext}} \Phi$ and $X_4^{\text{ext}} X_4^{\text{ext}} \Phi$ equal to three constants c_1 , c_2 , c_3 and eliminate x_u and U to yield x implicitly as

$$F(x, u, v, c_1, c_2, c_3) = 0$$

So x , that is σ , is finally determined.

Example 6 *Let us do this calculation for our running example*

$$\begin{aligned} \xi_{xx} - 4 \frac{\xi_x}{x} + 6 \frac{\eta}{x y} + 6 \frac{\xi}{x^2} \\ \eta_y + \xi_x - 4 \frac{\eta}{y} - 2 \frac{\xi}{x} \\ \eta_x - \frac{y \xi_x}{x} + 3 \frac{\eta}{x} + 2 \frac{y \xi}{x^2} \\ \xi_y + \frac{x \xi_x}{y} - \frac{\xi}{y} \end{aligned}$$

We will learn in the next subsection how to compute the invariants j , k , Ω , h_1 , h_2 from the coefficients of this Janet-base, so let me just state that

$$\begin{aligned} j &= 0 \\ k &= -\frac{x}{y} \\ \Omega &= 0 \\ h_1 &= \frac{2}{x} \\ h_2 &= \frac{4}{y} \end{aligned}$$

so that $U(x, y)$ is determined by

$$\begin{aligned} U_x &= \frac{1}{2} U^2 \\ U_y &= -\frac{x}{2y} U^2 - \frac{1}{y} U \end{aligned}$$

Making the substitution $x = x_1$, $y = 1 + x_1 y_1$ yields

$$\begin{aligned} U_{x_1} - \frac{y_1}{x_1} U_{y_1} - \frac{1}{2} U^2 &= 0 \\ \frac{1}{x_1} U_{y_1} + \frac{1}{2} \frac{x_1}{1 + x_1 y_1} U^2 + \frac{1}{1 + x_1 y_1} U &= 0 \end{aligned}$$

which gives the Riccati-equation in x_1 alone with y_1 as parameter

$$2(x_1 y_1 + 1) U_{x_1} - U^2 + 2y_1 U = 0$$

with solution

$$U(x_1, y_1) = 2 \frac{y_1}{4c_1 y_1^2 x_1 + 4c_1 y_1 + 1}$$

where the c_1 must be chosen such a function of y_1 that for $x_1 = 0$ the solution ceases to be dependent on y_1 and becomes a constant. If we call this new constant d_1 , then c_1 must be

$$c_1 = -\frac{1}{4} \frac{-2y_1 + d_1}{d_1 y_1}$$

Putting this in the solution gives

$$U(x_1, y_1) = -2 \frac{d_1}{-2x_1 y_1 + x_1 d_1 - 2}$$

Making the backsubstitution $x_1 = x$ and $y_1 = (y - 1)/x$ yields

$$U(x, y) = -2 \frac{d_1}{-2y + x d_1}$$

as general solution for the simultaneous Riccati-equations in x and y . Solving for d_1 gives

$$d_1 = 2 \frac{U y}{U x + 2}$$

and therefore

$$\Phi(U, x, y) = 2 \frac{U y}{U x + 2}$$

so that $2\Phi_U \sigma_x$ becomes

$$8 \frac{\sigma_x y}{(U x + 2)^2}$$

and $4\Phi_{UU} \sigma_x^2 + 4\Phi_U \sigma \sigma_x$ becomes

$$16 \frac{\sigma_x y (-2x \sigma_x + \sigma U x + 2\sigma)}{(U x + 2)^3}$$

Setting the last three expressions equal to c_1 , c_2 , c_3 and eliminating U and σ_x leaves us with

$$\sigma = \frac{1 - 2x c_2^2 - 2c_3 y + c_3 c_1 x}{2(-2y + c_1) c_2}$$

Setting $c_1 = 0$, $c_2 = 1$, $c_3 = 0$ gives

$$\sigma = \frac{1}{2} \frac{x}{y}$$

and this was indeed the substitution for x used to produce our Janet-base. Now we can put this value for σ into

$$\begin{aligned} h_1 &= 2 \frac{\rho_x}{\rho} - \frac{\sigma_{xx}}{\sigma_x} \\ h_2 &= 2 \frac{\rho_y}{\rho} - \frac{\sigma_{yy}}{\sigma_y} \end{aligned}$$

and with the values $h_1 = 2/x$ and $h_2 = 4/y$ from above get the equations

$$\begin{aligned} (\log \rho)_x &= \frac{2}{x} \\ (\log \rho)_y &= \frac{2}{y} \end{aligned}$$

which lead to

$$\log \rho = 2(\log x + \log y + c)$$

that is

$$\rho = c_1 x^2 y^2$$

which was our substitution for y .

Remark 9 *Regarding the constants that appeared in our solution for σ and ρ , it is indeed no loss of generality to specialize them in an arbitrary way. And it is secondly no chance, that the number (4) of constants is equal to the dimension of the lav $p, xp, yq, x^2p + xyq$ that we stated to be the inverse lav of our symmetry lav $g_{3,13}$.*

In fact these inverse lav consists of those infinitesimal point-transformations that transform the normal form of $g_{3,13}$ into itself and to choose the constants in our solution differently is identical with composing our original transformation which we wanted to reconstruct with one of these infinitesimal automorphisms. This statement, while being not obvious at this point, is a consequence of the way, LIE proved his main theorem about the invariants, theorem 5.

6.4 The use of non-linear Janet-algorithms in transforming the basic system

Let the general setup be as in subsection 6.2. .

By theorem 5 certain expressions in σ, ρ (LIEs invariants) and their derivatives can be expressed as functions of x, y alone if one uses the equations of the nonlinear system of pde (23) as "relations".

To utilize the equations of the system

$$a_{\alpha_{ij}}(x, y) = a'_{\alpha_{ij}}(\sigma, \rho, \sigma_x, \dots) \quad (28)$$

it would be only necessary to compute a Janet-base of this system with respect to any termorder in the functions σ, ρ and their derivatives. Then one could reduce the invariants, expressed with σ, ρ and their derivatives by this Janet-base. For the purpose of such a computation one would treat the left sides of (28) as indeterminate functions of x and y , and so compute once and for all the expression of the invariant as a rational function in the coefficients of a propounded Janet-base.

Unfortunately, the computation of a Janet-base, as described in section 2 of this paper is only defined for a system of *linear* pde.

Nevertheless it turns out, that by modification of only a few decisive steps in the algorithms of section 2 it is possible to generalize them for nonlinear pde, which are polynomial in the dependent variables.

Lemma 4 *Let e be a pde in the independents x^i and dependents u^j and let a termorder be introduced on the u^j . Then e can be written as*

$$e = c(u_J^j)^n + d$$

where c contains only monomials lower than u_J^j and d may in addition to these contain $(u_J^j)^l$, but with l less than n . c is the leading coefficient, u_J^j the leading monomial, $(u_J^j)^n$ the leading monomial power, $c(u_J^j)^n$ the leading term and d the reductum

Now we can define a new procedure for reduction of a pde e by a pde f

Algorithm 5 *write*

$$f = c(u_L^k)^m + d$$

as in lemma 4

set M to a list of all monomials u_J^i appearing in e

set N to the empty list.

```

repeat
  if  $M$  is part of  $N$  then return  $e$ 
  let  $u_J^i$  be the largest monomial in  $M$  (by rank)
  set  $M = M - u_J^i$ 
  compute  $a, n$  from  $e = a (u_J^i)^n + b$ 
  compute  $K$  with  $D_K u_L^k = u_J^i$ 
  if none such exists then start loop over
  if  $J = L$  then
    if  $n \geq m$  then
      let  $e = ce - a (u_L^k)^{n-m} f$ 
      compute new  $M$  from new  $e$ 
      start loop over
    else
      start loop over
  fi
else
  let  $f_1 = D_K f$ 
  let  $f_1 = c_1 u_J^k + d_1$  be the splitting as in lemma 4
  if  $c_1 = 0$  then add  $u_J^i$  to  $N$ , start loop over
  let  $e = c_1 e - a (u_J^k)^{n-1} f_1$ 
  compute new  $M$  from new  $e$ 
  start loop over
fi
end repeat

```

The other algorithm to be generalized is the computation of integrability conditions. So let

$$\begin{aligned} e &= a (u_J^i)^m + b \\ f &= c (u_K^i)^n + d \end{aligned}$$

be two pde with the same function in the leading monomial *and with no leading monomial a derivative of the other*. This latter condition can be assumed to be fulfilled when the integrability conditions are computed for elements from an autoreduced system (see [2] for details). The conditions entail like in section 2 that there are minimal L, M not of order zero with $D_L u_J^i = D_M u_K^i = u_N^i$. Write

$$\begin{aligned} D_L e &= a_1 u_N^i + b_1 \\ D_M f &= c_1 u_N^i + d_1 \end{aligned}$$

Then define

$$\text{IGB}(e, f) = a_1 D_M f - c_1 D_L e$$

as the integrability condition of e and f .

Only these two algorithms need to be replaced in the algorithms of section 2 to get an algorithm that computes a normal form for a system of nonlinear pde.

Example 7 We will now consider our running example of the lav $g_{3,13,ffff}$ with pattern Janet-base

$$\begin{aligned} &\xi_{xx} + a_1 \xi_x + a_2 \eta + a_3 \xi \\ &\eta_y + b_1 \xi_x + b_2 \eta + b_3 \xi \\ &\eta_x + c_1 \xi_x + c_2 \eta + c_3 \xi \\ &\xi_y + d_1 \xi_x + d_2 \eta + d_3 \xi \end{aligned}$$

The system (28) reduces by our nonlinear normal form algorithm to

$$\begin{aligned}
\frac{\partial}{\partial y} \sigma &= \frac{\frac{\partial}{\partial x} \sigma}{c_1} \\
\frac{\partial}{\partial y} \rho &= \frac{c_1 \left(\frac{\partial}{\partial x} \rho \right) + \rho c_3 + \rho c_1 a_1 + 2 \rho c_1^2 d_3}{c_1^2} \\
\frac{\partial^2}{\partial x^2} \sigma &= \frac{\left(\frac{\partial}{\partial x} \sigma \right) \left(c_1 \left(\frac{\partial}{\partial x} \rho \right) + \rho c_3 \right)}{c_1 \rho} \\
\frac{\partial^2}{\partial x^2} \rho &= \frac{1}{2} (\rho^2 c_3^3 - \rho^2 c_1^3 \left(\frac{\partial^2}{\partial x^2} a_1 \right) - 2 \rho^2 c_1^3 d_3 \left(\frac{\partial^2}{\partial x^2} c_1 \right) \\
&\quad + \rho^2 c_1 \left(\frac{\partial^2}{\partial x^2} c_1 \right) c_3 - \rho^2 c_1^2 \left(\frac{\partial^2}{\partial x^2} c_3 \right) - \rho^2 c_1^3 a_2 c_3 \\
&\quad + \rho^2 c_1^3 a_3 a_1 - 3 \rho^2 c_1^2 \left(\frac{\partial}{\partial x} c_3 \right) a_1 + 2 \rho \left(\frac{\partial}{\partial x} \rho \right) c_1^2 a_1 c_3 \\
&\quad - 4 \rho^2 c_1^3 d_3 c_3 a_1 + 3 \rho^2 c_1 \left(\frac{\partial}{\partial x} c_1 \right) c_3 a_1 \\
&\quad - 6 \rho^2 c_1^3 d_3 \left(\frac{\partial}{\partial x} c_1 \right) a_1 + 2 \rho \left(\frac{\partial}{\partial x} \rho \right) c_1 c_3^2 + 2 \rho^2 c_1^4 a_3 d_3 \\
&\quad - 4 \rho^2 c_1^3 \left(\frac{\partial}{\partial x} d_3 \right) \left(\frac{\partial}{\partial x} c_1 \right) + 2 \rho^2 c_1 \left(\frac{\partial}{\partial x} c_3 \right) \left(\frac{\partial}{\partial x} c_1 \right) \\
&\quad - 12 \rho^2 c_1^5 d_3^2 a_1 - 6 \rho^2 c_1^4 a_1^2 d_3 - 4 \rho^2 c_1^4 d_3^2 c_3 \\
&\quad - 6 \rho^2 c_1^4 \left(\frac{\partial}{\partial x} d_3 \right) a_1 - 12 \rho^2 c_1^4 d_3^2 \left(\frac{\partial}{\partial x} c_1 \right) \\
&\quad - 6 \rho^2 c_1^4 \left(\frac{\partial}{\partial x} a_1 \right) d_3 + \rho^2 c_1 a_1 c_3^2 - 12 \rho^2 c_1^5 \left(\frac{\partial}{\partial x} d_3 \right) d_3 \\
&\quad - \rho^2 c_1^2 a_1^2 c_3 + 2 \rho^2 c_1^2 d_3 \left(\frac{\partial}{\partial x} c_1 \right) c_3 - 2 \rho^2 c_1^5 a_2 d_3 \\
&\quad - 3 \rho^2 c_1^3 \left(\frac{\partial}{\partial x} a_1 \right) a_1 - 2 \rho^2 c_1^2 \left(\frac{\partial}{\partial x} a_1 \right) c_3 \\
&\quad - 6 \rho^2 c_1^3 \left(\frac{\partial}{\partial x} c_3 \right) d_3 + 2 \rho^2 c_1^2 d_3 c_3^2 - \rho^2 c_1^4 a_2 a_1 \\
&\quad - \rho^2 c_1^3 a_1^3 + 4 \rho \left(\frac{\partial}{\partial x} \rho \right) c_1^3 d_3 c_3 - 8 \rho^2 c_1^6 d_3^3 \\
&\quad + 2 \rho^2 \left(\frac{\partial}{\partial x} c_1 \right) c_3^2 - 2 \rho^2 \left(\frac{\partial}{\partial x} c_1 \right)^2 c_3 - 2 \rho^2 c_1^4 \left(\frac{\partial^2}{\partial x^2} d_3 \right) \\
&\quad + 3 \left(\frac{\partial}{\partial x} \rho \right)^2 c_1^3 a_1 + 3 \left(\frac{\partial}{\partial x} \rho \right)^2 c_3 c_1^2 + 6 \left(\frac{\partial}{\partial x} \rho \right)^2 c_1^4 d_3 \\
&\quad + \rho^2 c_1^2 a_3 c_3 - 2 \rho^2 c_1 \left(\frac{\partial}{\partial x} c_3 \right) c_3 - 4 \rho^2 c_1^3 \left(\frac{\partial}{\partial x} d_3 \right) c_3) / (\\
&\quad c_1^2 (c_1 a_1 + c_3 + 2 c_1^2 d_3) \rho)
\end{aligned}$$

This is an autoreduced system for which the integrability conditions are fulfilled, and therefore it can be used to reduce any expression in σ , ρ and their derivatives, as are the invariants Ω , k etc. introduced above.

Doing these reductions for our invariants we get their expressions in the coefficients of the pattern Janet-base:

$$\begin{aligned}
\Omega = & -\frac{1}{2}(2c_1^2 d_3 \left(\frac{\partial}{\partial x} c_1\right) c_3 + 4c_1^3 \left(\frac{\partial}{\partial x} d_3\right) c_3 \\
& + 2c_1^5 a_2 d_3 + 6c_1^4 \left(\frac{\partial}{\partial x} d_3\right) a_1 + 12c_1^4 d_3^2 \left(\frac{\partial}{\partial x} c_1\right) \\
& + 6c_1^4 \left(\frac{\partial}{\partial x} a_1\right) d_3 + 3c_1^3 \left(\frac{\partial}{\partial x} a_1\right) a_1 + 2c_1^2 \left(\frac{\partial}{\partial x} a_1\right) c_3 \\
& + 12c_1^5 \left(\frac{\partial}{\partial x} d_3\right) d_3 + c_1^3 a_2 c_3 + c_1^4 a_2 a_1 \\
& + 4c_1^3 \left(\frac{\partial}{\partial x} d_3\right) \left(\frac{\partial}{\partial x} c_1\right) - 2c_1 \left(\frac{\partial}{\partial x} c_3\right) \left(\frac{\partial}{\partial x} c_1\right) + 12c_1^5 d_3^2 a_1 \\
& + c_1^2 a_1^2 c_3 + 6c_1^4 a_1^2 d_3 + 4c_1^4 d_3^2 c_3 + c_1^2 \left(\frac{\partial}{\partial x} c_3\right) a_1 \\
& + 2c_1^3 d_3 \left(\frac{\partial^2}{\partial x^2} c_1\right) - c_1 \left(\frac{\partial^2}{\partial x^2} c_1\right) c_3 + 2c_1^3 \left(\frac{\partial}{\partial x} c_3\right) d_3 + c_1^3 a_1^3 \\
& + 4c_1^3 d_3 c_3 a_1 - c_1 \left(\frac{\partial}{\partial x} c_1\right) c_3 a_1 + 6c_1^3 d_3 \left(\frac{\partial}{\partial x} c_1\right) a_1 \\
& + 8c_1^6 d_3^3 + c_1^3 \left(\frac{\partial^2}{\partial x^2} a_1\right) + 2c_1^4 \left(\frac{\partial^2}{\partial x^2} d_3\right) + 2\left(\frac{\partial}{\partial x} c_1\right)^2 c_3 \\
& + c_1^2 \left(\frac{\partial^2}{\partial x^2} c_3\right) - 2c_1^4 a_3 d_3 - c_1^3 a_3 a_1 - c_1^2 a_3 c_3 \Big/ (c_1^2 \\
& (c_1 a_1 + c_3 + 2c_1^2 d_3))
\end{aligned}$$

$$k = \frac{1}{c_1}$$

$$\begin{aligned}
j = & -(2c_1^2 d_3 \left(\frac{\partial}{\partial x} c_1\right) c_3 + 4c_1^3 \left(\frac{\partial}{\partial x} d_3\right) c_3 + 2c_1^5 a_2 d_3 \\
& + 6c_1^4 \left(\frac{\partial}{\partial x} d_3\right) a_1 + 12c_1^4 d_3^2 \left(\frac{\partial}{\partial x} c_1\right) + 6c_1^4 \left(\frac{\partial}{\partial x} a_1\right) d_3 \\
& + 3c_1^3 \left(\frac{\partial}{\partial x} a_1\right) a_1 + 2c_1^2 \left(\frac{\partial}{\partial x} a_1\right) c_3 + 12c_1^5 \left(\frac{\partial}{\partial x} d_3\right) d_3 \\
& + c_1^3 a_2 c_3 + c_1^4 a_2 a_1 + 4c_1^3 \left(\frac{\partial}{\partial x} d_3\right) \left(\frac{\partial}{\partial x} c_1\right) \\
& - 2c_1 \left(\frac{\partial}{\partial x} c_3\right) \left(\frac{\partial}{\partial x} c_1\right) + 12c_1^5 d_3^2 a_1 + c_1^2 a_1^2 c_3 \\
& + 6c_1^4 a_1^2 d_3 + 4c_1^4 d_3^2 c_3 + c_1^2 \left(\frac{\partial}{\partial x} c_3\right) a_1 \\
& + 6c_1^3 d_3 \left(\frac{\partial^2}{\partial x^2} c_1\right) + c_1 \left(\frac{\partial^2}{\partial x^2} c_1\right) c_3 + 2c_1^3 \left(\frac{\partial}{\partial x} c_3\right) d_3 + c_1^3 a_1^3 \\
& + 4c_1^3 d_3 c_3 a_1 - c_1 \left(\frac{\partial}{\partial x} c_1\right) c_3 a_1 + 6c_1^3 d_3 \left(\frac{\partial}{\partial x} c_1\right) a_1 \\
& + 8c_1^6 d_3^3 + c_1^3 \left(\frac{\partial^2}{\partial x^2} a_1\right) + 2c_1^4 \left(\frac{\partial^2}{\partial x^2} d_3\right) - 2\left(\frac{\partial}{\partial x} c_1\right)^2 c_3 \\
& + c_1^2 \left(\frac{\partial^2}{\partial x^2} c_3\right) - 2c_1^4 a_3 d_3 - c_1^3 a_3 a_1 - c_1^2 a_3 c_3 \\
& - 4\left(\frac{\partial}{\partial x} c_1\right)^2 c_1 a_1 - 8\left(\frac{\partial}{\partial x} c_1\right)^2 c_1^2 d_3 + 2\left(\frac{\partial^2}{\partial x^2} c_1\right) c_1^2 a_1 \Big/ (\\
& c_1^3 (c_1 a_1 + c_3 + 2c_1^2 d_3))
\end{aligned}$$

$$h_1 = -\frac{c_3}{c_1}$$

$$h_2 = \frac{c_1 a_1 + 2 c_1^2 d_3 + \left(\frac{\partial}{\partial x} c_1\right) + \left(\frac{\partial}{\partial y} c_1\right) c_1}{c_1^2}$$

With these formulas it is now perfectly clear, how we could obtain the values of these invariants as they were presented in the example of the last subsubsection from the coefficients of the given Janet-base.

7 Application to the solution of ODEs

7.1 Determining the normal form of the ODE from knowledge of the normal form transformation

Given an ode $\omega(y^{(n)}, \dots, y', y, x) = 0$ (where $n = 2$ and $n = 3$ will be the cases that we take into consideration) we can now determine the type of its symmetry lav $\mathcal{L}$ and we can determine the transformation $x = \sigma(x, y)$, $y = \rho(x, y)$ that leads from the coordinates of the normal form of the symmetry lav to the coordinates in which the ode $\omega = 0$ is given.

We will now determine the ode ω_0 in the coordinates of the normal form lav by transformation of which our given ode ω results.

As we know from section 4.2 ω_0 is a function

$$\omega_0 = F(\phi_1, \phi_2, \phi_2, \dots)$$

of differential invariants of the normal form lav (which we will refer to as $\mathcal{L}_0$ in the following).

In general $\phi_1, \phi_2, \dots$ are quasilinear differential expressions

$$\phi_1 = y^{(k)} - g_1(y^{(k-1)}, \dots, y, x) \quad (29)$$

$$\phi_2 = y^{(k+1)} - g_2(y^{(k)}, \dots, y, x) \quad (30)$$

$$\phi_3 = y^{(k+2)} - g_3(y^{(k+1)}, \dots, y, x) \quad (31)$$

...

where the k depends on the lav. Which ϕ_i actually can appear in the normal form ω_0 for our ode ω is determined by the order of ω , only those need to be taken into consideration, which have an order not greater than ω .

To fix ideas let these be ϕ_1 and ϕ_2 . Then we apply our known transformation $x = \sigma(x, y)$, $y = \rho(x, y)$ to the right sides of the equations (29) and (30) finding thereby the expressions of ϕ_1 and ϕ_2 in the coordinates in which ω is expressed. The new right sides will be quasilinear again, so that we can write

$$\phi_1 = y^{(k)} - h_1(y^{(k-1)}, \dots, y, x) \quad (32)$$

$$\phi_2 = y^{(k+1)} - h_2(y^{(k)}, \dots, y, x) \quad (33)$$

Solving these for $y^{(k)}$ and $y^{(k+1)}$ respectively, we get equations

$$y^{(k)} = \phi_1 + h_1(y^{(k-1)}, \dots, y, x)$$

$$y^{(k+1)} = \phi_2 + h_2(y^{(k)}, \dots, y, x)$$

...

Consider now in the two last equations ϕ_1 and ϕ_2 just as symbols and substitute for $y^{(k)}$ and $y^{(k+1)}$ accordingly in ω . An equation

$$\omega(y^{(k+1)}, \dots, y', y, x) = \omega_0(\phi_1, \phi_2, y', y, x)$$

results, which is the searched for expression of ω in terms of invariants ϕ_1 and ϕ_2 .

Remark 10 In general the expression ω_0 will not depend on ϕ_1 and ϕ_2 only, but on other remaining derivatives of y and on x too. Those result from systems of imprimitivity of the symmetry lav.

7.2 A concrete example

In the case of our example ode

$$\begin{aligned} & 2y^8x^4\left(\frac{\partial^2}{\partial x^2}y\right) + (-2y^6x^5 + 2x^3)\left(\frac{\partial}{\partial x}y\right)^3 \\ & + (-2y^7x^4 - 6yx^2)\left(\frac{\partial}{\partial x}y\right)^2 + (4y^8x^3 + 6y^2x)\left(\frac{\partial}{\partial x}y\right) \\ & - 2y^3 \end{aligned} \quad (34)$$

with transformation

$$x = \frac{1}{2}x/y, \quad y = \frac{1}{4}x^2y^2 \quad (35)$$

from normal coordinates to actual coordinates, the only differential invariant that comes into consideration is

$$\phi_1 = \left(\frac{\partial^2}{\partial x^2}y\right)y - \frac{1}{2}\left(\frac{\partial}{\partial x}y\right)^2$$

This equation is transformed into

$$\begin{aligned} 0 &= y^8x^4\left(\frac{\partial^2}{\partial x^2}y\right) + (\phi_1x^3 - y^6x^5)\left(\frac{\partial}{\partial x}y\right)^3 \\ &+ (-3\phi_1x^2y - y^7x^4)\left(\frac{\partial}{\partial x}y\right)^2 \\ &+ (3\phi_1xy^2 + 2y^8x^3)\left(\frac{\partial}{\partial x}y\right) - \phi_1y^3 \end{aligned}$$

by (35). Solved for y'' it reads as

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= (-\phi_1x^3\left(\frac{\partial}{\partial x}y\right)^3 + y^6x^5\left(\frac{\partial}{\partial x}y\right)^3 + 3\phi_1\left(\frac{\partial}{\partial x}y\right)^2x^2y \\ &+ y^7x^4\left(\frac{\partial}{\partial x}y\right)^2 - 3\phi_1x\left(\frac{\partial}{\partial x}y\right)y^2 - 2y^8x^3\left(\frac{\partial}{\partial x}y\right) \\ &+ \phi_1y^3)/(y^8x^4) \end{aligned}$$

Substituting this into (34) and factorizing yields

$$2\left(\left(\frac{\partial}{\partial x}y\right)x - y\right)^3(\phi_1 - 1)$$

Now we have only to solve $\phi_1 - 1 = 0$ and transform the solutions

$$x = 2\frac{\sqrt{-2+y-C1}}{-C1} - C2, \quad x = -2\frac{\sqrt{-2+y-C1}}{-C1} - C2$$

by (35) to get

$$\begin{aligned} 0 &= \frac{1}{16}(4y^4x^2-C1 + (-4-C2^2-C1^2-32)y^2 \\ &- 4x-C1^2-C2y - x^2-C1^2)/(y^2-C1) \end{aligned}$$

from the first solution and a similar expression for the second.

Additional solutions result from the systems of imprimitivity and can be found in the same way, starting with the solutions of $y'x - y$.

Maybe at this point the question arises, in how far we have simplified the integration problem aside from having it simplified from (34) to the more concise $\phi_1 - 1$.

But indeed the simplification goes much further, because for every second order ode, polynomial in y'' , y' , y , and having symmetry lav of type $_{3,13}$ the normal form will be (up to soi factors) a polynomial in ϕ_1 with constant coefficients.

So we just need to solve the odes

$$\phi_1^n + a_1\phi_1^{n-1} + \cdots + a_0 = 0 \quad (36)$$

which is the same as solving

$$\phi_1 = c$$

for arbitrary c (root of the polynomial (36)). This can be done in advance for arbitrary c .

References

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