A short proof for Giambelli's formula

Jürgen Böhm*

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Abstract

Giambelli's formula allows to write a Schubert class $\sigma_{a_1,\ldots,a_k}$ as a determinant $\det(\sigma_{a_i+j-i})_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant k}}$.

It can be proved using only Pieri's formula as a prerequisite, but known proofs tend to have a combinatorial setup that is not well to memorize.

The following article gives a compact and well memorizable proof making use of a convenient auxiliary notation.

1 Statement and Proof

The task of the Schubert calculus is to calculate products of the form

$$\sigma_{a_1,\ldots,a_k}\cdot\sigma_{b_1,\ldots,b_k}=\sum\Gamma^{(c_i)}_{(a_i),(b_i)}\sigma_{c_1,\ldots,c_k} \tag{1}$$

where $n-k\geqslant a_1\geqslant a_2\geqslant\cdots\geqslant a_k\geqslant 0$ and similar for the $b_1,\ldots,b_k$ and $c_1,\ldots,c_k$. The $\Gamma^{(c_i)}_{(a_i),(b_i)}$ are integers, the so called Littlewood-Richardson coefficients.

For more details see Eisenbud and Harris [1, Chapter 4] or Fulton [2, Section 14.7].

Further on we also write

$$[a_1,\ldots,a_k][b_1,\ldots,b_k]$$

for such a product to ease notation. Also $[a_1,\ldots,a_s]$ is written for

$$[a_1,\ldots,a_s,0,\ldots,0]$$

*jboehm@gmx.net

with $s < k$.

The first base of such a calculation is the *Formula of Pieri*

$$[a] \cdot [b_1, \ldots, b_k] = \sum_{\substack{|c|=|b|+a \\ b_i \leqslant c_i \leqslant b_{i-1}}} [c_1, \ldots, c_k] \tag{2}$$

where $|c| = c_1 + \cdots + c_k$ and correspondingly $|b| = b_1 + \cdots + b_k$.

If one could write every $[a_1, \ldots, a_k]$ as a sum of products $[e_1][e_2] \cdots [e_k]$ one could compute all $[a_1, \ldots, a_k][b_1, \ldots, b_k]$ by Pieri's formula.

This decomposition is the result of the *Formula of Giambelli.*

It states that

$$\begin{aligned}
\sigma_{a_1,\ldots,a_k} &= \det(\sigma_{a_i-i+j})_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant k}} = \\
&= \det \begin{pmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+k-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+k-2} \\ \sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & \cdots & \sigma_{a_3+k-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{a_k-k+1} & \sigma_{a_k-k+2} & \sigma_{a_k-k+3} & \cdots & \sigma_{a_k} \end{pmatrix}
\end{aligned} \tag{3}$$

For a typical combinatorial proof see Griffiths and Harris [3, p. 205].

These formula can be proven from that of Pieri by developing the determinant.(It is $\sigma_a = 0$ for $a < 0$).

In detail:

$$\begin{aligned}
[a_1, a_2, a_3, a_4, \ldots] = [a_1]&[a_2, a_3, a_4, \ldots] \\
&- [a_2 - 1][a_1 + 1, a_3, a_4, \ldots] \\
&+ [a_3 - 2][a_1 + 1, a_2 + 1, a_4, \ldots] \\
&- [a_4 - 3][a_1 + 1, a_2 + 1, a_3 + 1, a_5, \ldots] + - \cdots
\end{aligned} \tag{4}$$

by developing the Giambelli-matrix along the first column and using induction on Giambelli's formula itself.

By a simple combinatorial consideration we get the relations

$$[a_1][a_2, a_3, a_4, \ldots] = [a_1, a_2, a_3, \ldots] + [a_2 - 1][a_1 + 1, a_3, a_4, \ldots]_{2;a_2} \tag{5}$$

and

$$\begin{aligned}
[a_2 - 1][a_1 + 1, a_3, a_4, \ldots] = [a_2 - 1]&[a_1 + 1, a_3, a_4, \ldots]_{2;a_2} + \\
&+ [a_3 - 2][a_1 + 1, a_2 + 1, a_4, \ldots]_{3;a_3}
\end{aligned} \tag{6}$$

and

$$[a_3 - 2][a_1 + 1, a_2 + 1, a_4, \ldots] = [a_3 - 2][a_1 + 1, a_2 + 1, a_4, \ldots]_{3;a_3} + \\ + [a_4 - 3][a_1 + 1, a_2 + 1, a_3 + 1, a_5, \ldots]_{4;a_4} \quad (7)$$

and further

$$[a_4 - 3][a_1 + 1, a_2 + 1, a_3 + 1, a_5, \ldots] = \\ [a_4 - 3][a_1 + 1, a_2 + 1, a_3 + 1, a_5 \ldots]_{4;a_4} + \\ + [a_5 - 4][a_1 + 1, a_2 + 1, a_3 + 1, a_4 + 1, a_6, \ldots]_{5;a_5} \quad (8)$$

and so on till the end of the development of the determinant in the k-th row.

Notation 1.1 *Here*

$$[a][b_1, b_2, b_3, \ldots]_{i;a_i}$$

stands for the distributions of a over $[b_1, b_2, b_3, \ldots]$ *(as in the formula of Pieri) for which* b_i *will be filled at most to* a_i *(including).*

References

[1] David Eisenbud and Joe Harris. *3264 and all that. A second course in algebraic geometry.* Cambridge: Cambridge University Press, 2016.

[2] William Fulton. *Intersection theory. 2nd ed.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 2. Berlin: Springer. xiii, 470 p. , 1998.

[3] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* A Wiley-Interscience Publication. New York etc.: John Wiley & Sons. XII, 813 p., 1978.