INVARIANTS VIA KERNEL OF KODAIRA-SPENCER-MAP

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ABSTRACT. In an algebraic family $F_a(x_i, t_j) = 0$ isomorphism invariants in the parameters t_j are found as solutions of LPDE calculated from the kernel of the Kodaira-Spencer map.

1. Kodaira-Spencer

If $V(F_a) = F \subseteq \mathbb{P}^m_{k[t_j]}$ is projective over $T = \mathrm{Spec}\,(k[t_j])$, thus the $F_a(x_i, t_j)$ are homogeneous in the x_i, then one has a Kodaira-Spencer map

$$K_t : T_t T \mapsto H^1(F_t, \mathcal{T}_{F_t}) = T_{F_t}(\mathcal{M})$$

where $\mathcal{M}$ is the moduli space of the F_t.

For every algebraic invariant $u_s(t_j)$ which comes from $T \to \mathcal{M}$, one has

$$\ker du_s(t_j)|_t \supseteq \ker K_t.$$

The system of linear subspaces $\ker K_t$ defines a system of linear PDEs on (t_j) whose solutions are the algebraic invariants $u_s(t_j)$ that we sought above.

2. Kodaira-Spencer example

Let $T = \mathrm{Spec}\,(k[a, b]) = \mathrm{Spec}\,(A)$ and $P = \mathbb{P}^2_T$ with $P = \mathrm{proj}\,(A[X, Y, Z])$, and further $X = V(F) \subseteq P$ with $F = F(a, b, X, Y, Z)$, for example

$$F = Y^2 Z - (X^3 + aXZ^2 + bZ^3)$$

We have the sequence

$$0 \to \pi^*\Omega_T \to \Omega_X \to \Omega_{X|T} \to 0$$

and the dualization $\mathcal{H}om(-, \mathcal{O}_X)$

$$0 \to \mathcal{T}_{X|T} \to \mathcal{T}_X \to \pi^*\mathcal{T}_T \to 0 \tag{1}$$

The diagram for the fiber at $t \in T$ is

$$\begin{tikzcd} X \arrow[d, "\pi"] & X_t \arrow[l, "j"'] \arrow[d, "\pi'"] \\ T & t \arrow[l, "i"] \end{tikzcd}$$

Applying $j^*(-)$ to the sequence (1), we obtain

$$0 \to \mathcal{T}_{X_t|t} \to \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t} \to \mathcal{T}_{T,t} \otimes \mathcal{O}_{X_t} \to 0$$

The long exact sequence gives

$$H^0(\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t}) \xrightarrow{\psi_t} \mathcal{T}_{T,t} \otimes H^0(\mathcal{O}_{X_t}) \xrightarrow{K_t} H^1(\mathcal{T}_{X_t|t}) \tag{2}$$

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where K_t is the Kodaira-Spencer map and ψ gives a surjection onto $\ker K_t$.

Thus, if we want to determine $\ker K_t$ as $\operatorname{im}\psi_t$, we must describe $\mathcal{T}_X$ explicitly: At first it is

$$0 \to \mathcal{T}_X \to \mathcal{T}_P \otimes_{\mathcal{O}_P} \mathcal{O}_X \to \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \tag{3}$$

where $\mathcal{I}$ is the ideal sheaf $F\mathcal{O}_P$.

Furthermore, $P = T \times_k \mathbb{P}^2_k$ and hence

$$\Omega_P = p_1^*\Omega_T \oplus p_2^*\Omega_{\mathbb{P}^2_k} \tag{4}$$

With the Euler sequence $0 \to \Omega_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2} \to 0$ and its dualization we obtain

$$\begin{aligned} (5)\quad & \mathcal{T}_P = p_1^*\mathcal{T}_T \oplus p_2^*\mathcal{T}_{\mathbb{P}^2_k} = \\ &= \mathcal{O}_P\partial_a + \mathcal{O}_P\partial_b + (\mathcal{O}_P(1)\partial_X + \mathcal{O}_P(1)\partial_Y + \mathcal{O}_P(1)\partial_Z)/\mathcal{O}_P(X\partial_X + Y\partial_Y + Z\partial Z)\end{aligned}$$

With this representation by explicit vector fields, from the sequence (3) we obtain for $H^0(\mathcal{T}_X)$ the representation as the kernel of the map

$$v \in H^0(\mathcal{T}_P \otimes \mathcal{O}_X) \mapsto v \vdash F \tag{6}$$

and concretely we have

$$\begin{aligned} (7)\quad \mathcal{T}_P \otimes \mathcal{O}_X = (&\mathcal{O}_X\partial_a + \mathcal{O}_X\partial_b + \\ &\mathcal{O}_X(1)\partial_X + \mathcal{O}_X(1)\partial_Y + \mathcal{O}_X(1)\partial_Z)/\mathcal{O}_X(X\partial_X + Y\partial_Y + Z\partial_Z)\end{aligned}$$

We can write this as

$$0 \to \mathcal{L} \to \mathcal{W} \to \mathcal{T}_P \otimes \mathcal{O}_X \to 0 \tag{8}$$

and then consider the diagram from (3)

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & H^0_{\geqslant 0}(\mathcal{W})/H^0_{\geqslant 0}(\mathcal{L}) \arrow[r, "v\vdash F"] & R(d) \\ 0 \arrow[r] & K' \arrow[r] \arrow[u] & H^0_{\geqslant 0}(\mathcal{W}) \arrow[r, "v\vdash F"] \arrow[u] & R(d) \arrow[u, "="'] \end{tikzcd} \tag{9}$$

Then $\widetilde{K} = \mathcal{T}_X$ and $(K^{\mathrm{sat}})_0 = H^0(\widetilde{K}) = H^0(\mathcal{T}_X)$.

The projections of K_0 and K_0' onto $A\partial_a + A\partial_b$ are identical, so that instead of determining K_0 we may also use the simpler K_0', where no quotient has to be taken.

From the sequence

$$0 \to \mathcal{O}_P(-d+e) \xrightarrow{F} \mathcal{O}_P(e) \to \mathcal{O}_X(e) \to 0 \tag{10}$$

we obtain here, with $d = 3$ and $e = 0, 1$, the equalities

(11)

$$H^0(\mathcal{O}_X) = H^0(\mathcal{O}_P) = A = k[a,b], \quad H^0(\mathcal{O}_X(1)) = H^0(\mathcal{O}_P(1)) = XA + YA + ZA$$

Thus the vector fields $v \in H^0(\mathcal{W})$ are always of the form

$$\begin{aligned} (12)\quad v = &A\partial_a + A\partial_b + \\ &+ (AX + AY + AZ)\partial_X + (AX + AY + AZ)\partial_Y + (AX + AY + AZ)\partial_Z\end{aligned}$$

or, written with functions,

$$v = w_a(a,b)\partial_a + w_b(a,b)\partial_b + \\ + w_X(a,b;X,Y,Z)\partial_X + w_Y(a,b;X,Y,Z)\partial_Y + w_Z(a,b;X,Y,Z)\partial_Z \tag{13}$$

where the w_X, w_Y, w_Z are linear and homogeneous of degree 1 in X, Y, Z.

These elements v which, when applied to F, annihilate it, i.e. give $v \vdash F = 0$, are, projected onto $A\partial_a + A\partial_b = \mathfrak{T}_T$, precisely the kernels of the Kodaira-Spencer maps K_t.

We thus obtain these w_a, w_b as parts of the solution of the system of linear PDEs

$$v \vdash F = 0 \tag{14}$$

$$(w_a)_X, (w_a)_Y, (w_a)_Z = 0 \tag{15}$$

$$(w_b)_X, (w_b)_Y, (w_b)_Z = 0 \tag{16}$$

$$w_X = Xw_X^X(a,b) + Yw_X^Y(a,b) + Zw_X^Z(a,b) \tag{17}$$

$$w_Y = Xw_Y^X(a,b) + Yw_Y^Y(a,b) + Zw_Y^Z(a,b) \tag{18}$$

$$w_Z = Xw_Z^X(a,b) + Yw_Z^Y(a,b) + Zw_Z^Z(a,b) \tag{19}$$

$$(w_X^X)_X, (w_X^X)_Y, (w_X^X)_Z = 0 \tag{20}$$

$$\cdots \tag{21}$$

$$(w_Z^Z)_X, (w_Z^Z)_Y, (w_Z^Z)_Z = 0 \tag{22}$$

One can try, for example, to solve this system with a computer algebra system; the system *Maple* finds the desired solution in the example, equation (23).

Of course one does not really need to solve a system of LPDEs here; one simply takes all $w_a, w_b, w_X^X, \ldots, w_Z^Z$ to be functions of a, b and computes in $v \vdash F = 0$ all coefficients $c_{ijk}(a,b)$ of $X^iY^jZ^k$. They form a system of syzygy equations for the $w_a, w_b, w_X^X, \ldots, w_Z^Z$, which can be solved by the usual Gröbner basis methods.

One obtains (in one way or another):

$$(w_a\partial_a + w_b\partial_b)G = 2a\partial_a G + 3b\partial_b G = 0 \tag{23}$$

as the condition for an invariant $G = G(a,b)$. The general solution of this condition equation is

$$G(a,b) = g(\frac{b}{a^{3/2}}) \tag{24}$$

and indeed the known j-invariant

$$j = 6912\frac{a^3}{4a^3 + 27b^2} \tag{25}$$

can be expressed rationally in terms of the invariant b^2/a^3.

3. Kodaira–Spencer in general

In general, $X \subseteq \mathbb{P}_T^n = P$ with $T = \mathrm{Spec}\,(k[t_j]) = \mathrm{Spec}\,(A)$ and $X = V(F_a(X_i, t_j))$ with $a = 1, \ldots, s$. Then

$$S = A[X_0, \ldots, X_n], \quad I = (F_1, \ldots, F_s), \quad R = S/I \tag{26}$$

and

$$P = \mathrm{proj}\,(S), \quad X = \mathrm{proj}\,(R). \tag{27}$$

For the sake of correctness, we assume that X is nonsingular, so that the sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{P|k} \otimes \Omega_X \to \Omega_{X|k} \to 0 \tag{28}$$

is an exact sequence of sheaves.

We again use the sequence (1), which now forces us to compute the expression $H^0(\mathcal{T}_X)$ in (3).

We therefore write (3) as

$$0 \to N \to (\sum_j R\partial_{t_j} + \sum_i R(1)\partial_{X_i})/R(\sum_i X_i\partial_{X_i}) \xrightarrow{\delta} \mathrm{Hom}(I/I^2, R) \tag{29}$$

and note that, upon applying $\widetilde{(-)}$, the sheafification, one recovers the sequence (3). In particular, for

$$D_P = (\sum_j R\partial_{t_j} + \sum_i R(1)\partial_{X_i})/R(\sum_i X_i\partial_{X_i})$$

one always has $\widetilde{D}_P = \mathcal{T}_P \otimes \mathcal{O}_X$.

Now

$$\phi : M = \sum_\nu R(-d_\nu)e_\nu \to I/I^2 \tag{30}$$

with $\phi(e_\nu) = F_\nu + I^2$ and $d_\nu = \deg_X F_\nu$ is a surjection, so that we can also rewrite (29) as

$$0 \to N \to (\sum_j R\partial_{t_j} + \sum_i R(1)\partial_{X_i})/R(\sum_i X_i\partial_{X_i}) \xrightarrow{\delta'} \mathrm{Hom}(M, R) \tag{31}$$

where

$$\delta(\partial_{t_j}) \vdash \phi(e_\nu) = \delta'(\partial_{t_j}) \vdash F_\nu = \partial_{t_j} F_\nu \tag{32}$$

$$\delta(\partial_{X_i}) \vdash \phi(e_\nu) = \delta'(\partial_{X_i}) \vdash F_\nu = \partial_{X_i} F_\nu \tag{33}$$

hold.

Thus we have clarified all terms and maps in the sequence (31) and can, using the usual techniques provided by Macaulay2, apply $H^0(\mathrm{sheaf}(-))$ to the module sequence (31) and obtain $H^0(T_X) = H^0(\widetilde{N})$ explicitly as a submodule of

$$T_P = H^0(\widetilde{D}_P). \tag{34}$$

Subsequently, we simply project the generators of $H^0(\widetilde{N})$ onto $H^0(\sum_j R\partial_{t_j})$, just as we did in the special case of a single hypersurface.

To make this computationally possible, that is to compute $H^0(\widetilde{N})$ and the projection to A^m we use the relation

$$\mathrm{Hom}_R(B^q, N)_0 = H^0(X, \widetilde{N}) \tag{35}$$

with the irrelevant ideal of R, that is $B = (X_0, \dots, X_n) \subseteq R$

We introduce the map

$$p_N : N \to R^m = R\,\partial_{t_1} + \cdots + R\,\partial_{t_m}, \tag{36}$$

which is the projection to the first m components, the space where the sought for vector-fields live, and apply to it the functor $\mathrm{Hom}_R(B^q, -)$ to get

$$p_H : \mathrm{Hom}_R(B^q, N) \to \mathrm{Hom}_R(B^q, R^m). \tag{37}$$

Now let $G = \mathrm{Hom}_R(B^q, N)_0$ a system of A-generators of the 0-components with the projection map $p_G : G \to \mathrm{Hom}_R(B^q, N)$.

Let further be $i : B^q \to R$ the inclusion and

$$\eta : \mathrm{Hom}_R(R, R^m) \to \mathrm{Hom}_R(B^q, R^m) \tag{38}$$

the application of $\mathrm{Hom}_R(-, R^m)$ to i.

All together we have the diagram

$$\begin{tikzcd} \mathrm{Hom}_R(R, R^m) \arrow[r, "\eta"] & \mathrm{Hom}_R(B^q, R^m) \\ G \arrow[u, "u"] \arrow[ur, "w"] \arrow[r, "p_G"] & \mathrm{Hom}_R(B^q, N) \arrow[u, "p_H"'] \end{tikzcd} \tag{39}$$

The map u delivers us the geneators of the sought for vector-fields in $A^m = A\partial_{t_1} + \cdots + A\partial_{t_m}$, which ultimately come from $H^0(P, \widetilde{N}) = H^0(P, \mathcal{T}_X)$.

4. True invariants

Investigating the classical Legendre family

$$F(x, y, z, c) = y^2 z - x(x - z)(x - cz) \tag{40}$$

we get from the Kodaira-Spencer method no restricting LPDE, that is c is already considered invariant, which is clear because infinitesimally neighboured c have non-isomorphic fibers.

But we know, that the true invariant is

$$j = \frac{256\left(c^2 - c + 1\right)^3}{c^2\left(c - 1\right)^2} \tag{41}$$

To give the KS-method a little bit more "elbow space" we introduce a second parameter d to get

$$F(x, y, z, c, d) = y^2 z - x(x - z)(x - cz) + dz^3 \tag{42}$$

This leads to the LPDE:

$$4\left(c^2 - c + 1\right)\left(\frac{\partial}{\partial c}G(c, d)\right) - 2\left(c^2 - 6cd - c + 3d\right)\left(\frac{\partial}{\partial d}G(c, d)\right) \tag{43}$$

Its general solution is

$$G(c, d) = f\left(\frac{2c^3 - 3c^2 - 3c + 27d + 2}{27\left(c^2 - c + 1\right)^{3/2}}\right) \tag{44}$$

Squaring and putting $d = 0$ leads to

$$j_1 = \frac{\left(c - 2\right)^2\left(2c - 1\right)^2\left(1 + c\right)^2}{729\left(c^2 - c + 1\right)^3} \tag{45}$$

By eliminating c from j and j_1 we get the rational relation

$$j = \frac{6912}{4 - 729 j_1}. \tag{46}$$

One could ask, if this is the hint of a general method to get "true" invariants from "near misses" where the parameters are non-isotrivial in every infinitesimal direction, but are mapped finitely to the true moduli space.